Prediction - Core; Nash Bargaining

14.04 Intermediate Micro Theory: Lecture 15

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The theory to be reviewed in this section was proposed by Edgeworth (1881). His aim was to explain how the presence of many interacting competitors would lead to the emergence of a system of prices taken as given by economic agents, and consequently to a Walrasian equilibrium outcome. Edgeworth’s work had no immediate impact. The modern versions of his theory follow the rediscovery of his 

The theory of the core is distinguished by its parsimony. Its conceptual apparatus does not appeal to any specific trading mechanism nor does it assume any particular institutional setup. Informally, the notion of competition that the theory explores is one in which traders are well informed of the characteristics (endowments and preferences) of other traders, and in which the members of any group of traders can bind themselves to any mutually advantageous agreement. The simplest example is a buyer and a seller exchanging a good for money, but we can also have more complex arrangements involving many individuals and goods.
Formally, we consider an economy with $I$ consumers. Every consumer $i$ has consumption set $\mathbb{R}_+^L$, and endowment vector $\omega_i \geq 0$, and a continuous, strictly convex, strongly monotone preference relation $\succeq_i$. There is also a publicly available constant returns convex technology $Y \subset \mathbb{R}_+^L$. For example, we could have $Y = -\mathbb{R}_+^L$, that is, a pure exchange economy. All of these assumptions are maintained for the rest of the section.

As usual, we say that an allocation $x = (x_1, \ldots, x_I) \in \mathbb{R}_+^{LI}$ is feasible if $\sum_i x_i = y + \sum_i \omega_i$ for some $y \in Y$.

With a slight abuse of notation, we let the symbol $I$ stand for both the number of consumers and the set of consumers. Any nonempty subset of consumers $S \subset I$ is then called a coalition. Central to the concept of the core is the identification of circumstances under which a coalition of consumers can reach an agreement that makes every member of the coalition better off. Definition 18.B.1 provides a formal statement of these circumstances.
Definition 18.B.2: We say that the feasible allocation \( x^* = (x^*_1, \ldots, x^*_I) \in \mathbb{R}_+^I \) has the core property if there is no coalition of consumers \( S \subset I \) that can improve upon \( x^* \). The core is the set of allocations that have the core property.

We can see in the Edgeworth box of Figure 18.B.1 that for the case of two consumers the core coincides with the contract curve. With two consumers there are only three possible coalitions: \( \{1, 2\} \), \( \{1\} \), and \( \{2\} \). Any allocation that is not a Pareto optimum will be blocked by coalition \( \{1, 2\} \). Any allocation in the Pareto set that is not in the contract curve will be blocked by either \( \{1\} \) or \( \{2\} \). With more than two consumers there are other potential blocking coalitions, but the fact that the coalition of the whole is always one of them means that all allocations in the core are Pareto optimal.
Definition 18.B.1: A coalition $S \subset I$ improves upon, or blocks, the feasible allocation $x^* = (x_1^*, \ldots, x_i^*) \in \mathbb{R}^L_i$ if for every $i \in S$ we can find a consumption $x_i \geq 0$ with the properties:

(i) $x_i \succeq_i x_i^*$ for every $i \in S$.
(ii) $\sum_{i \in S} x_i \in Y + \{\sum_{i \in S} \omega_i\}$.

Definition 18.B.1 says that a coalition $S$ can improve upon a feasible allocation $x^*$ if there is some way that, by using only their endowments $\sum_{i \in S} \omega_i$ and the publicly available technology $Y$, the coalition can produce an aggregate commodity bundle that can then be distributed to the members of $S$ so as to make each of them better off.

Figure 18.B.1
The core equals the contract curve in the two-consumer case.
Pareto Optimum and the Core

Figure 18.B.2
An allocation in the contract curve that can be blocked with two replicas.
Proposition 18.B.2: Denoting by $hn$ the $n$th individual of type $h$, suppose that the allocation

$$x^* = (x_{11}^*, \ldots, x_{1n}^*, \ldots, x_{1N}^*, \ldots, x_{H1}^*, \ldots, x_{Hn}^*, \ldots, x_{HN}^*) \in \mathbb{R}_+^{LHN}$$

belongs to the core of the $N$-replica economy. Then $x^*$ has the equal-treatment property, that is, all consumers of the same type get the same consumption bundle:

$$x_{hn}^* = x_{hn}^*$$

for all $1 \leq m, n \leq N$ and $1 \leq h \leq H$. 

Proposition 18.B.3: If the feasible type allocation $x^* = (x_1^*, \ldots, x_N^*) \in \mathbb{R}_+^L$ has the core property for all $N = 1, 2, \ldots$, that is, $x^* \in C_N$ for all $N$, then $x^*$ is a Walrasian equilibrium allocation.
“Indeed, the influence of an individual participant on the economy cannot be mathematically negligible, as long as there are only finitely many participants. Thus *a mathematical model appropriate to the intuitive notion of perfect competition must contain infinitely many participants*. We submit that the most natural model for this purpose contains a continuum of participants, similar to the continuum of points on a line or the continuum of particles in a fluid." (Robert J. Aumann, 1964, “Markets with a Continuum of Traders", *Econometrica*).

“One day I received in the mail an article written by Milnor and Shapley – an analysis of voting in a situation in which there are some large voters and what they called an "ocean" of small voters...Then in the summer of 1961 there was a conference on Recent Advances in Game Theory at Princeton University. Herb Scarf gave a paper there that was a forerunner of the Debreu-Scarf paper on the core of an economy, and an outgrowth of previous work by Shubik (and by Edgeworth). Scarf’s model had a denumerable infinity of traders, divided into a finite number of types, and he got an equivalence theorem between the core and the competitive equilibrium. However the model had various defects...I remembered the paper by Milnor and Shapley about "oceanic" games when hearing Scarf’s model and said to myself, “surely, the continuum just *has* to be the right way of doing that."" (Robert J. Aumann, "Interview with Feiwel", 1987, *Arrow and the Ascent of Modern Economic Theory*)
The Continuum Economy

Robert J. Aumann (1964) broached the equivalence question and provided the proof. (Karl Vind (1964), almost simultaneously, proved the same result, albeit via different means). However, their theorems employed a mathematical method previously (largely) unseen in economics: measure theory. This required a translation of traditional economic concepts – agents, coalitions, allocations, equilibrium, etc. – into this new mathematical language.

Not everyone has been happy with this transformation. Koopmans (1974) considered it a "fanciful extension". The discomfort was caused by a crucial feature of Aumann’s model: the assumption that there was an uncountably infinite number of agents in the economy. Intuitively, this means that Aumann assumed that there as many agents as there are points on a line, a continuum of people.

Firstly, as Aumann would perhaps be the first to admit, there is mathematical convenience in a continuum. There are ready-made theorems and tools we can use here. Secondly, the concept of an "atomless" economy – an economy without "big players" – can be made precise in a continuum economy. This, he argues, gives precision to the meaning of "perfect" competition in manner which any other number of agents cannot.

We have an infinite number of households/agents. Let us define H as the set of households. A "coalition", as we know, is a group of agents, and thus a subset of H. A $\sigma$—algebra is merely a collection of subsets of H which have certain properties.
Let the interval $[0, 1]$ be our "set of people", $H$. Note that this is a continuum: there are as many people in the economy as there are points on a line in the $[0, 1]$ segment, an uncountably infinite number of people. A set of "names" with no numerical meaning.

$S$ is "bigger" than $T$, then we would like it that $\mu(S) > \mu(T)$, i.e. the measure of $S$ is greater than the measure of $T$.

Then a coalition is merely a "subset" of $[0, 1]$ and the natural "measure" of this subset is length.

What if a coalition has no "length"? For instance, what happens when there is a coalition $S$ formed by five people, each in a different place on the $[0, 1]$ segment? Well, here we begin to see the magic of the continuum: a person or any finite group of people have no length and thus are of measure zero, i.e. $\mu(S) = 0$. 

![Fig. 1 - Set of Agents and a Coalition](image-url)
Computing the core: Telser (1994)

- Economy consists of two airlines and three travelers
- First airline can carry up to two passengers at a total cost of 85
- Second airline can carry up to three passengers at a total cost of 150
- Payoff of airline $y_i = \text{revenue minus cost (or 0 if no flight)}$
- Travelers’ willingness to pay: $B_1 = 55$, $B_2 = 60$, $B_3 = 70$
- Payoff of traveler $x_i = \text{willingness to pay minus cost (or 0 if no travel)}$
- 5 agents $\implies 2^5 - 1 = 31$ coalitions
Consider proposed core allocation \((x_1, x_2, x_3, y_1, y_2)\)

Net value of a flight with passengers \(D \subseteq I\) on airline \(j\),
\[
V_{D,j} = \sum_{i \in D} WTP_i - C_j
\]

If \(\sum_i x_i + y_j < V_{D,j}\) and \(|D| \leq \text{capacity}_j\), then \(D \cup \{j\}\) blocks

Size 1 coalitions: \(x_i \geq 0, y_i \geq 0\) (can choose not to travel or fly airplane)

Size 2 coalitions: cannot create positive net value from travel with 2 agents, so no new restrictions that are not already implied from size 1 coalitions
Size 3 coalitions: new restriction $x_1 + x_2 + y_1 \geq 30$

($= V_{\{1,2\},1} = 55 + 60 - 85$)

Similarly, also have $x_1 + x_3 + y_1 \geq 40$, $x_2 + x_3 + y_1 \geq 45$; note that cannot create positive net value from travel on airline 2 without all three agents

Size 4 coalitions: new restriction $x_1 + x_2 + x_3 + y_2 \geq 35$

Size 5 coalition: $x_1 + x_2 + x_3 + y_1 + y_2 = 45$ ($V_{\{2,3\},1} = 45$ is the net value of the Pareto optimal allocation)

Can show that no allocation satisfies all of these restrictions, i.e. core is empty
Bargaining Game

The bargaining game [edit source]

The **bargaining game** or **Nash bargaining game** is a simple two-player game used to model bargaining interactions. In the Nash bargaining game, two players demand a portion of some good (usually some amount of money). If the total amount requested by the players is less than that available, both players get their request. If their total request is greater than that available, neither player gets their request. A **Nash bargaining solution** is a (Pareto efficient) solution to a Nash bargaining game. According to Walker (2005), Nash's bargaining solution was shown by John Harsanyi to be the same as Zeuthen's solution of the bargaining problem (*Problems of Monopoly and Economic Warfare*, 1930).

**Nash bargaining solution** [edit source]

John Nash proposed that a solution should satisfy certain axioms:

1. Invariant to affine transformations or Invariant to equivalent utility representations
2. Pareto optimality
3. Independence of irrelevant alternatives
4. Symmetry

Let $u$ and $v$ be the utility functions of Player 1 and Player 2, respectively. In the **Nash bargaining solution**, the players will seek to maximize $(u(x) - u(d)) \times (v(y) - v(d))$, where $u(d)$ and $v(d)$, are the **status quo** utilities (i.e., the utility obtained if one decides not to bargain with the other player). The product of the two excess utilities is generally referred to as the **Nash product**. Intuitively, the solution consists of each player getting her status quo payoff (i.e., noncooperative payoff) in addition to an equal share of the benefits accruing from cooperation (Muthoo 1999, pp. 15–16).

Impossibility of a Walrasian Bargaining Solution
Sertel & Yıldız (2002)

Abstract

Is there a bargaining solution that pays out the Walrasian welfare for exchange economies? We show that there is none, for there are distinct exchange economies whose Walrasian equilibrium welfare payoffs disagree but which define the same bargaining problem and should have hence determined the same bargaining solution and its payoffs.

Keywords: Walrasian equilibrium, bargaining, implementation via bargaining.

JEL Classification Numbers: 026, 0211.
The first order conditions of this problem are

\[
\max_{c(s)} \prod_i \left\{ \mathbb{E}_s \left\{ \zeta_i u_i[c(s)] + (1 - \zeta_i) u_i[y_i] \right\} - \mathbb{E}_{y_i} [u_i(y_i)] \right\}^{\alpha_i}
\]

subject to \( \sum_i \zeta_i c_i(s) \leq \sum_i \zeta_i y_i \), for some vector \( \alpha \) such that \( \sum_i \alpha_i = 1 \) and \( \alpha_i \geq 0 \).

\[\sqrt{\frac{\alpha_i}{\mathbb{E}_s \zeta_i [u_i(c_i(s)) - u_i(y_i)]}} u'(c_i(s)) = q(s)\]

where, again, \( q(s) \) are the Lagrange multipliers of the resource constraints. Therefore, the equivalent Pareto weight in the Planner’s problem is exactly

\[\lambda_i = \frac{\alpha_i}{\mathbb{E}_s \zeta_i [u_i(c_i(s)) - u_i(y_i)]}\]
Lemma: If an allocation \( x = (x_1, \ldots, x_{HN}) \in \mathbb{R}^{LHN}_+ \) belongs to the core, then \( x^* \) has the equal treatment property: \( x_{hm} = x_{hn} \)

Proof: suppose not. For each type there is a worst-treated individual, say \( h_1 \). Let \( \hat{x}_h = (1/N) \sum_n x_{nh} \). Then \( \hat{x}_h \gtrless_h x_{h1} \), strict if type \( h \) does not exhibit equal treatment (by strict convexity). Coalition \( S = \{h1\}_{h=1}^H \), with consumption \( (\hat{x}_1, \ldots, \hat{x}_h) \). Feasibility of \( x \) implies 
\[
\sum_h \hat{x}_h = \frac{1}{N} \sum_n x_{hn} = \frac{1}{N} (\sum_h N\omega_h) = \sum_h \omega_h.
\]

Implication: we can restrict to type allocations \( x = (x_1, \ldots, x_H) \in \mathbb{R}^{LH}_+ \)