Fundamental Welfare Theorems
14.04 Intermediate Micro Theory: Lecture 15

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Outline

- Competitive equilibria are Pareto optimal
- Any Pareto Optimum supported as equilibrium with Transfers
- Sufficient Assumption and proofs
- Finite dimensional Euclidean Space and Valuation Equilibria in more general spaces
Dynamics of the income process

Basic picture: Inequality and uncorrelated shocks.
Recall: Households are of very different size.

(a) Comovement of household incomes (deviation from village average) Aurepalle.
Dynamics of individual consumption

Basic picture: consumption profiles much smoother than income process

(a) Comovement of household consumptions (grain only) (deviation from village average) Aurepalle.
<table>
<thead>
<tr>
<th>Variable</th>
<th>Aurepalle</th>
<th>Shirapur</th>
<th>Kanzara</th>
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<tbody>
<tr>
<td><strong>Coefficient</strong></td>
<td>$R^2$</td>
<td>Pr &gt; $F$</td>
<td>$R^2$</td>
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<td>Area of Operated Landholdings</td>
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<td>Value of Owned Bulls</td>
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<td>0.5485</td>
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<td>Value of Inheritance</td>
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<td>Number of Married Sons</td>
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<td>Number of Migrants</td>
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<td>Total Wealth</td>
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<td>0.4826</td>
<td>0.0113*</td>
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<td>Age of Head of Household</td>
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<td>13.6948</td>
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<tr>
<td>Age of Head squared</td>
<td>0.0223</td>
<td>-0.1484</td>
<td>-0.1484</td>
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TABLE XI

Relationship Between Intercepts and Wealth (All Consumption)
First Welfare Theorem

Let \((x^*, y^*, p, w)\) be a price equilibrium with transfers. Then, if preferences are rational and locally non-satiated, then \((x^*, y^*)\) is a Pareto Optimal allocation.

Note that for this theorem, we need not assume anything about the consumption set \(X_i\) for each agent, other that it has to be consistent with the requirement of local non satiation (i.e. for any given bundle there must exist an arbitrarily close bundle that is strictly preferred).

We do not have to assume anything about the production sets either, other than the implicit assumption that \(Y_j \neq \emptyset\) for all \(j\) (because we know there exist some \(y^*\) that is part of a price equilibrium with transfers).
Statement of the Theorem

Graphical Representation of FWT

Figure: Fi Theorem

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Lemma

To prove this theorem, we need the following Lemma

Lemma

If preferences are locally non-satiated and $x_i^*$ is maximal for $\succeq_i$ in the budget set $B = \{ x_i \in X_i : p_i x_i \leq w_i \}$, then $x_i \succeq_i x_i^*$ implies $p_i x_i \geq p_i x_i^*$.
Using this lemma, we can now prove the First Welfare Theorem: that is, that any price equilibrium with transfers gives a Pareto Optimal allocation.

Proof.

Suppose, by contradiction, that there exists some other feasible allocation \((x, y)\) such that \(x_i \succeq_i x_i^*\) for all \(i\) and there exist \(i' : x_{i'} \succ_{i'} x_{i'}^*\). We must have that \(p x_{i'} > p x_{i'}^*\) and, using the previous Lemma, we must also have that \(p x_i \geq p x_i^*\). Therefore

\[
\sum_{i=1}^{l} p x_i > \sum_{i=1}^{l} p x_i^* \tag{1}
\]
Proof of FWT II

Because $y_i^*$ maximizes profits given prices, we must have that for all $j$:

$$py_j^* \geq py_j$$

which implies that:

$$\sum_{j=1}^{J} py_j^* \geq \sum_{j=1}^{J} py_j$$  \hspace{1cm} (2)

Putting together (1) and (2), and using the feasibility of $(x^*, y^*)$:

$$\sum_{i=1}^{I} px_i > \sum_{i=1}^{I} px_i^* = p\overline{w} + \sum_{j=1}^{J} py_j^* \geq p\overline{w} + \sum_{j=1}^{J} py_j \quad \Longrightarrow \quad (3)$$

$$p \left( \sum_{i=1}^{I} x_i - \overline{w} - \sum_{j=1}^{J} y_j \right) > 0 \quad \Longrightarrow \quad \sum_{i=1}^{I} x_i \neq \overline{w} + \sum_{j=1}^{J} y_j$$

So $(x, y)$ was not feasible.
Claim: Any target Pareto Optimal Allocation can be implemented as a competitive equilibrium.

It is easy to see that at any Pareto allocation \( (x^*_A, 1), (x^*_A, 2), (x^*_B, 1), (x^*_B, 2) \) we must have that

\[
\text{Slope of indiff. curve at } (x^*_A, 1), (x^*_A, 2) = \text{Slope of indiff. curve at } (x^*_B, 1), (x^*_B, 2) 
\]

\[
\iff RMS^A (x^*_A, 1), (x^*_A, 2) \equiv \frac{\partial u_A/\partial x_1}{\partial u_A/\partial x_2} (x^*_A, 1), (x^*_A, 2) = RMS^B (x^*_B, 1), (x^*_B, 2) 
\]

and define

\[ \gamma = RMS^A (x^*_A, 1), (x^*_A, 2) = RMS^B (x^*_B, 1), (x^*_B, 2) \]  \hspace{1cm} (1)
Non-affordable Pareto Optimal Allocation
If prices were $p = (\gamma, 1)$, then the budget constraint would have exactly the same slope as the two indifference curves, and both agents would choose their demands at $(x_{A,1}^*, x_{A,2}^*)$ and $(x_{B,1}^*, x_{B,2}^*)$ respectively. The only problem is that with those prices, the value of their endowment may not be sufficient to purchase the target allocation. For example, the endowment point is such that, at prices $p = (\gamma,1)$ agent $B$ does not have sufficient income to buy it:

$$\gamma x_{B,1}^* + x_{B,2}^* > \gamma \omega_{B,1} + \omega_{B,2}$$

so $\gamma$ could not be an equilibrium price. However, if we were allowed to make transfers between agents, we could “tax” agent $A$ by giving her a negative transfer, and giving the revenue to agent $B$

$$t_A \equiv \gamma \omega_{B,1} + \omega_{B,2} - \gamma x_{B,1}^* - x_{B,2}^* < 0 \quad \quad t_B \equiv \gamma x_{B,1}^* + x_{B,2}^* - \gamma \omega_{B,1} - \omega_{B,2} = -t_A > 0$$

since the initial Pareto allocation $\left((x_{A,1}^*, x_{A,2}^*), (x_{B,1}^*, x_{B,2}^*)\right)$ is feasible. With such transfers, is easy to see that this allocation would in fact be a price equilibrium with transfers! This construction can be replicated for all Pareto optimal allocations: i.e. by making transfers, any target Pareto optimal allocation can be implemented as a competitive equilibrium.
Second Welfare Theorem
Take an economy \( \mathcal{E} = \left\{ \{X_i, u_i(\cdot), \omega_i\}_{i=1}^l, \{Y_j\}_{j=1}^J, \{\theta_{ij}\} \right\} \) such that:

1. \( X_i \subseteq \mathbb{R}_+^l \) are convex, open sets, for all \( i = 1, 2, \ldots, l \)
2. \( Y_j \subseteq \mathbb{R}^L \) are convex and closed sets, and admit a concave transformation function \( F_j : \mathbb{R}^L \rightarrow \mathbb{R} \); i.e.
   \[
   Y_j = \{ y \in \mathbb{R}^L : F_j(y) \geq 0 \} \quad (13)
   \]
3. Preferences are given by concave and locally non-satiated utility functions \( u_i : X_i \rightarrow \mathbb{R} \) (equivalent to preferences being rational, convex, continuous and locally non-satiated)
4. There exist \( (\tilde{x}, \tilde{y}) \) such that \( \tilde{x} \in X_i, F_j(y_j) > 0 \) for all \( j = 1, \ldots, J \) and
   \[
   \sum_i \tilde{x}_i \ll \overline{\omega} + \sum_j \tilde{y}_j
   \]

**Theorem (Second Welfare Theorem)**

*Under assumptions 1 to 4, for any \( \lambda \in \Delta^l \) there exist a price vector \( p \in \mathbb{R}_+^L \) and a vector of wealth levels \( w^* \in \mathbb{R}_+^l \) such that \( (x^*, y^*, p, w^*) \) is a Walrasian equilibrium with transfers of \( \mathcal{E} \).*
Outline of the Proof

We will present the proof for when all functions are differentiable (that is, $u_i(\cdot)$ and $F_j(\cdot)$ are differentiable) and preferences are strictly monotonic. For the general proof, see Negishi (1960). Also, without loss of generality, we will assume $\lambda_i > 0$ for all $i$. (Note: $\lambda$ are Pareto weights.)

The proof will be lengthy, but constructive, and consists of 3 parts:

1. Characterize the solution to the Pareto Problem for a given vector $\lambda \in \Delta^I$.
2. Characterize the conditions for an equilibrium with transfers.
3. Make the mapping between the Pareto optimal allocation and the Walrasian Equilibrium with transfers that implements it, by identifying that under some conditions, the conditions for both allocations are identical.
First Part: Solving the Pareto Problem

\[
\max_{(x,y)} \sum_{i=1}^{I} \lambda_i u_i(x_i)
\]  
(14)

subject to

\[
\sum_{i=1}^{I} x_{i,l} \leq \bar{\omega}_l + \sum_{j=1}^{J} y_{j,l} \text{ for all } l = 1, \ldots, L
\]  
(15)

\[
F_j(y_j) \geq 0 \text{ for all } j = 1, \ldots, J
\]  
(16)

Pareto program satisfies the conditions of the Kuhn-Tucker Theorem, since:

- The objective function is concave for any \( \lambda \in \Delta^I \)
- Inequalities are given by \( L + J \) concave functions; namely
  \[
g_l(x, y) = \bar{\omega}_l + \sum_j y_{j,l} - \sum_i x_i \text{ and } g_j(x, y) = F_j(y)
\]
- The feasible set \( \mathcal{F} \) has non-empty relative interior, where

\[
\mathcal{F} = \left\{ (x, y) : \begin{cases} 
  x_i \in X_i & \text{for all } i = 1, \ldots, I \\
  F_j(y_j) \geq 0 & \text{for all } j = 1, \ldots, J \\
  \sum_{i=1}^{I} x_{i,l} \leq \bar{\omega}_l + \sum_{j=1}^{J} y_{j,l} & \text{for all } l = 1, \ldots, L
\end{cases} \right\}
\]  
(17)
Constrained Optimization – General Recipe

- Problem statement for optimization with inequality constraints:
  \[
  \max_x f(x) \\
  \text{s.t. } g_i(x) \geq 0 \text{ for } i = 1,\ldots,n
  \]
  
  where \( f \) and \( g_i \) are real-valued continuously differentiable functions

- How to solve?
- Step 1: Form the Lagrangian.
  \[
  L = f(x) + \sum_{i=1}^{n} \lambda_i g_i(x)
  \]

- Step 2: Write out the first-order conditions for the \( x_k \)s.
  \[
  \frac{\partial L}{\partial x_k} = \frac{\partial f}{\partial x_k} + \sum_{i=1}^{n} \lambda_i \frac{\partial g_i}{\partial x_k} = 0 \forall k
  \]
The Lagrangian for this program is:

\[
\mathcal{L}(x, y, \gamma, \phi) = \sum_{i=1}^{I} \lambda_i u_i(x_i) + \sum_{l=1}^{L} \gamma_l \left( \bar{w} + \sum_{j=1}^{J} y_j - \sum_{i=1}^{I} x_{i,l} \right) + \sum_{j=1}^{J} \phi_j F_j(y_j) \tag{18}
\]

Therefore, the Pareto optimal allocation will be characterized by the existence of Lagrange multipliers \(\gamma^*\) and \(\phi^*\) such that:

1. \(\nabla_{(x,y)} \mathcal{L} = 0\). This gives us two sets of equations:

\[
\frac{\partial \mathcal{L}}{\partial x_{i,l}} = 0 \iff \lambda_i \frac{\partial u_i}{\partial x_l}(x_i^*) = \gamma_l^*
\]

\[
\frac{\partial \mathcal{L}}{\partial y_{j,l}} = 0 \iff -\phi_j^* \frac{\partial F_j}{\partial y_l}(y_j^*) = \gamma_l^*
\]

(19)

2. Primal feasibility: just that \(F_j(y_j^*) \geq 0\) and \(\sum x_i^* \leq \bar{w} + \sum y_j^*\)

3. Dual feasibility: just that \(\gamma^*, \phi^* \geq 0\)

4. Complementary Slackness: for all \(l = 1, \ldots, L\):

\[
\gamma_l^* \left[ \sum_{i=1}^{I} x_{i,l}^* - \bar{w}_l - \sum_{j=1}^{J} y_{j,l}^* \right] = 0
\]

(20)
Now, since $u_i(\cdot)$ is strictly increasing, and at least one agent $\hat{i}$ must have $\lambda_{\hat{i}} > 0$ (if not, they could never sum up to 1), then we must have that $\gamma_{i}^* > 0$ for all $i$. Therefore, we can substitute condition 20 with the condition

$$\sum_{i=1}^{l} x_{i,l} = \bar{\omega}_l + \sum_{j=1}^{J} y_{j,l} \text{ for all } l = 1, \ldots, L$$

(21)

Using the Kuhn-Tucker conditions we derived, we have shown the following conclusion:

**Characterization of PO:** $(x^*, y^*) = (x^*(\lambda), y^*(\lambda))$ is the solution to 14 (and hence PO) if $(x^*, y^*)$ is feasible and $\exists \gamma^* \in \mathbb{R}^L_+ \text{ and } \phi \in \mathbb{R}_+^J$ such that $(x^*, y^*, \gamma^*, \phi^*)$ satisfy conditions 19, and 21.
Second Part: Characterizing Equilibrium with Transfers

As usual, \((x^*, y^*, p, w^*)\) is an equilibrium with transfers if and only if

1. Consumers optimize:
   \[
   x_i^* \in \arg\max_{x \in X_i} u_i(x) \quad \text{subject to} \quad px \leq w_i^* \quad \forall i = 1, 2, \ldots, I
   \]  
   (22)

2. Firms maximize profits:
   \[
   y_j^* \in \arg\max_{y \in \mathbb{R}^L} py \quad \text{subject to} \quad F_j(y) \geq 0 \forall j = 1, \ldots, J
   \]  
   (23)

3. Resource constraints are satisfied:
   \[
   \sum_{i=1}^{I} x_{i,l}^* = \bar{w}_l + \sum_{j=1}^{J} y_{j,l}^* \forall l = 1, 2, \ldots, L
   \]  
   (24)

4. Wealth levels are feasible:
   \[
   \sum_{i=1}^{I} w_i^* = \sum_{i=1}^{I} \left( p\omega_i + \sum_{j=1}^{J} \theta_{ij}p y_j^* \right)
   \]  
   (25)
For consumer optimization, we use the Kuhn-Tucker theorem on the maximization problem of 22, and conclude that $x^*_i$ satisfies $22 \iff \exists \mu_i \geq 0$ such that

$$\frac{\partial u_i}{\partial x_l} (x^*_i) = \mu_i p_l \text{ for all } l = 1, 2, ..., L$$  \hspace{1cm} (26)$$

and

$$px^* = w_i^*$$ \hspace{1cm} (27)$$

For profit maximization, we again use Kuhn-Tucker to conclude that $y^*_j$ satisfies $23 \iff \exists \delta_j \geq 0$ such that

$$-\delta_j \frac{\partial F_j}{\partial y_l} (y^*_j) = p_l \text{ for all } l = 1, 2, ..., L$$ \hspace{1cm} (28)$$

and

$$F_j (y^*_j) \geq 0$$ \hspace{1cm} (29)$$

**Characterization of Equilibria:** $(x^*, y^*, p, w^*)$ is an equilibrium with transfers $\iff (x^*, y^*)$ satisfies the RC 24, $w^*$ is feasible (25) and $\exists \mu \in \mathbb{R}^I_+$ and $\delta \in \mathbb{R}^J_+$ such that the consumer and the firms FOCs (26 and 28) are satisfied.
Third Part: Putting all the pieces together

Then, to show our desired result, we need to show the existence of

- a price vector $p$
- a wealth vector $w^*$
- Lagrange multipliers $\mu_i$ and $\delta_j$

such that $(x^*, y^*, p, w^*)$ satisfies all the above conditions. What are our candidates?

Let’s compare our FOCs of consumer utility, in both the Pareto Problem and in the walrasian equilibrium. According to 19 and 26 we get

$$\frac{\partial u_i}{\partial x^*_l}(x^*_i) = \frac{1}{\lambda_i}\gamma^*_i \text{ for all } i, l \quad \frac{\partial u_i}{\partial x^*_l}(x^*_i) = \mu_i p_l \text{ for all } i, l$$

Then, the guess for the price $p$, marginal utility of wealth $\mu_i$ and wealth $w^*_i$ that would implement the allocation $(x^*, y^*)$ is obvious:

$$p_l \equiv \gamma^*_l \quad \mu_i \equiv \frac{1}{\lambda_i} \quad w^*_i \equiv \sum_{l=1}^L \gamma^*_l x^*_{i,l}$$
The implementing price of commodity \( l \) is the \textit{shadow price of the resource constraint of that commodity}. Likewise, the marginal utility of wealth is the inverse of the Pareto Weight associated with the desired allocation \((x^*, y^*)\).

With this definition of prices and wealth, is clear that \( x_i^* \) maximizes her utility over her budget set. For any two agents \( i, h \) and commodities \( l, k \):

\[
RMS_{ik}^i (x_i^*) \equiv \frac{\partial u_i / \partial x_l}{\partial u_i / \partial x_k} (x_i^*) = \frac{(1 / \lambda_i) \gamma_i^*}{(1 / \lambda_i) \gamma_k^*} = \frac{(1 / \lambda_h) \gamma_i^*}{(1 / \lambda_h) \gamma_k^*} = RMS_{lk}^h (x_h^*)
\]

i.e. all agents equalize their marginal rates of substitution among themselves, and we pick the relative prices to equalize them!

Let’s compare the FOCs of the firm’s problem:

\[
- \phi_j^* \frac{\partial F}{\partial y_l} (y_j^*) = \gamma_j^* \quad \text{for all } l, j \quad \quad - \delta_j \frac{\partial F}{\partial y_l} (y_j^*) = p_l \quad \Rightarrow \quad \gamma_j^* \quad \text{for all } l, j
\]

by def.
	herefore setting

\[
\delta_j \equiv \phi_j^* \geq 0 \quad \text{for all } j
\]

makes \( y_j^* \) to be the solution of the profit maximization problem in 23.
Third Part: Putting all the pieces together

So far, we have shown that with prices \( p_i \equiv \gamma_i^* \) and wealth levels \( w_i^* \equiv \gamma^* x_i^* \) both consumers and agents. Resource feasibility is guaranteed by condition 21. Finally, we have that

\[
\sum_{i=1}^{l} w_i^* = \sum_{i=1}^{l} px_i^* = p \sum_{i=1}^{l} x_i^* = p \left( \sum_{i=1}^{l} \omega_i + \sum_{j=1}^{J} y_j^* \right) =
\]

bec. of RC

so 25 is also satisfied. Hence, we have finally shown that 
\( (x^*, y^*, p = \gamma^*, w^* = px^*) \) is a Walrasian Equilibrium with transfers, as we wanted to show.
In the First Welfare Theorem, we show that any equilibrium allocation is Pareto optimal. The Second Welfare Theorem proves a converse to the previous theorem: basically it shows that any Pareto optimal allocation can be achieved through a competitive equilibrium with transfers; i.e. appropriate redistribution of wealth.

**Price Quasi-Equilibrium with transfers**

An allocation \((x^*, y^*)\) and a price vector \(p \in \mathbb{R}^J_+\) constitute a **price quasi-equilibrium with transfers** if there exists an assignment of wealth levels \((w_1, w_2, \ldots, w_I)\) such that \(\sum_{i=1}^{I} w_i = p\bar{\omega} + \sum_{j=1}^{J} p y_j^*\) such that:

1. For each \(j = 1, 2, \ldots, J\), \(y_j^*\) maximizes profits given prices: \(p y_j^* \geq p y_j\) for all \(y_j \in Y_j\)
2. For every \(i = 1, 2, \ldots, I\), if \(x_i \succ_i x_i^*\) then \(p x_i \geq w_i\)
3. The allocation is feasible: \(\sum_{i=1}^{I} x_i^* = \bar{\omega} + \sum_{j=1}^{J} y_j^*\)
See that the only difference between a quasi-equilibrium and an equilibrium is given by condition 2. Essentially, if \( x_i \) is preferred to \( x_i^* \), it must cost at least as much. If \( w_i = px_i^* \) then \( x_i^* \) minimizes the expenditure \( px \) on the contour set \( V(x_i^*) \). Note that if instead it said "\( x_i \succ_i x_i^* \implies px_i > w_i \)" then this would mean that \( x_i^* \) is optimal in the budget set, since any other bundle that is better, could not be in the budget set.

**Theorem**

*Suppose \( Y_j \) is convex for all \( j = 1, 2, ..., J \) and the preference relations \( \succeq_i \) are rational, convex and satisfy local non-satiation. Then, for any Pareto-Optimal Allocation \( (x^*, y^*) \), there exist a price vector \( p \in \mathbb{R}^L_+ \) such that \( (x^*, y^*, p) \) is a price quasi-equilibrium with transfers.*

The idea of the theorem is to generalize the argument of the second welfare theorem for the 1x1 model. We will define an analog of the set \( V^* = \{ x : x \succeq x^* \} \) and the technology \( Y^* \), and find the implementing equilibrium prices by finding the separating hyperplane between both.
Supporting Hyperplane Theorem

Figure M.G.6
The supporting hyperplane theorem
Graphical Representation
Debreu (1954)

In this paper, Debreu shows the equivalence between a *valuation equilibrium* (defined below) and *Pareto optimum* allocations.

Importantly, he is able to show the welfare theorems for very general settings, where commodity spaces and production sets are general linear spaces (not necessarily finite dimensional).

This level of generality is useful to deal with infinite horizon economies or economies with lotteries (discussed below).

The environment can be described as:

- **Consumer** $i = 1, \ldots, I$ chooses a bundle $x_i \in X_i \subset L$ ($L$ is a linear space)
  - Preferences are indicated by a complete ordering on $X_i$, denoted by $\succeq_i$
- **Producer** $j = 1, \ldots, J$ chooses production plan $y_j \in Y_j \subset L$
- Market clearing is given by $x - y = \zeta$, where $x = \sum_{i=1}^{I} x_i$, $y = \sum_{j=1}^{J} y_j$, and $\zeta$ is the endowment

**State of the Economy**

A $(I + J)$–tuple $[(x_i), (y_j)]$ is called a *state of the economy*. A state $[(x_i), (y_j)]$ is called *attainable* if $x_i \in X_i$ for all $i$, $y_j \in Y_j$ for all $j$, $x - y = \zeta$
Valuation Equilibrium and Pareto Optimum

Consider the following definition (see SLP pp. 445):

**Linear functional**

A *linear functional* on a normed vector space \((S, \| \cdot \|)\) is a function \(\phi : S \to \mathbb{R}\) satisfying

\[
\phi(\alpha x + \beta y) = \alpha \phi(x) + \beta \phi(y), \text{ for all } x, y \in S, \text{ all } \alpha, \beta \in \mathbb{R}
\]  

Valuation Equilibrium

Let \(v(z)\) denote a linear mapping on \(L\). A state \([(x^0_i), (y^0_j)]\) is a *valuation equilibrium* with respect to \(v(z)\) if

1. \([(x^0_i), (y^0_j)]\) is attainable
2. For every \(i\), \(x_i \in X_i\), \(v(x_i) \leq v(x^0_i)\) implies \(x_i \leq x^0_i\)
3. For every \(j\), \(y_j \in Y_j\) implies \(v(y_j) \leq v(y^0_j)\)
A Valuation Equilibrium is a Pareto optimum

The following assumptions will be made:

1. For every $i$, $X_i$ is convex
2. For every $i$, $x_i' \in X_i$, $x_i'' \in X_i$, $x_i' < i x_i''$ implies $x_i' < i (1 - \alpha)x_i' + \alpha x_i''$, for all $\alpha$, $0 < \alpha < 1$

Theorem

Under assumption 1 and 2, every valuation equilibrium $[(x_i^0), (y_j^0)]$, where no $x_i^0$ is a saturation point, is a Pareto optimum
Infinite-Dimensional Commodity Spaces: Debreu (1954)

A Pareto Optimum is a Valuation Equilibrium

Let $x_i', x_i''$ be points of $X_i$, define $I(x_i', x_i'') = \{ \alpha | [(1 - \alpha)x_i' \in \alpha x_i'' \in X_i] \}$. Consider the following additional assumptions:

3. For every $i$, $x_i, x_i', x_i'' \in X_i$ the sets $\{ \alpha \in I(x_i', x_i'') | (1 - \alpha)x_i' + \alpha x_i'' \geq_i x_i \}$ and $\{ \alpha \in I(x_i', x_i'') | (1 - \alpha)x_i' + \alpha x_i'' \leq_i x_i \}$ are closed.

4. Let $Y = \sum_{j=1}^{J} Y_j$. $Y$ is convex.

5. $L$ is finite dimensional and/or $Y$ has an interior point.

Theorem

Under assumptions I-V, with every Pareto optimum $[(x_0^i), (y_0^j)]$, where some $x_0^i$ is not a saturation point, is associated a (nontrivial) continuous linear form $v(z)$ on $L$ such that:

1. For every $i$, $x_i \in X_i, x_i \geq_i x_0^i$ implies $v(x_i) \geq v(x_0^i)

2. For every $j$, $y_j \in Y_j$ implies for all $v(y_j) \leq v(y_0^j)$