Applications of Welfare Theorems in Hyperspace:

- Incentive constrained contracts
- The space of lotteries
- Welfare theorems extensions and qualifications
## CD Withdrawal Options

**Calculating effective return on a CD after paying early withdrawal**

<table>
<thead>
<tr>
<th>CD Term</th>
<th>2-Year</th>
<th>3-Year</th>
<th>4-Year</th>
<th>5-Year</th>
<th>7-Year</th>
</tr>
</thead>
<tbody>
<tr>
<td>APY</td>
<td>1.25%</td>
<td>2.25%</td>
<td>2.76%</td>
<td>3.00%</td>
<td>3.51%</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Withdraw after year</th>
<th>Effective Return</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.63%</td>
</tr>
<tr>
<td>2</td>
<td>1.12%</td>
</tr>
<tr>
<td>3</td>
<td>1.38%</td>
</tr>
<tr>
<td>4</td>
<td>1.50%</td>
</tr>
<tr>
<td>5</td>
<td>0.00%</td>
</tr>
<tr>
<td>6</td>
<td>0.00%</td>
</tr>
<tr>
<td>7</td>
<td>0.00%</td>
</tr>
</tbody>
</table>

Computations use simple interest (compounded annually).

Early withdrawal penalty is previous 6 months interest for 2, 3, 4, and 5 year CDs and last year’s interest for 7-Year CD.
## Pension Options

<table>
<thead>
<tr>
<th>Type</th>
<th>Description</th>
<th>Details</th>
</tr>
</thead>
<tbody>
<tr>
<td>Single Life Annuity</td>
<td>An annuity that pays you for your lifetime until you pass away.</td>
<td>Typically provides highest annuity benefit, but does not provide for any survivor benefits for a spouse.</td>
</tr>
<tr>
<td>Joint and Survivor Annuity</td>
<td>An annuity that pays you for your lifetime until you pass away. Payments will then continue for the life of your spouse.</td>
<td>This will provide lower initial monthly payments than a single life annuity because payments will continue to your surviving spouse.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>There are also typically survivor payout options (as a % of initial benefit amount) such as 100%, 75%, or 50%. Each option will lower the spousal annuity payment accordingly. A lower spousal survivor benefit will increase the initial annuity payment amount.</td>
</tr>
<tr>
<td>Period Certain Payment</td>
<td>An annuity payment that is guaranteed for a specific number of years, even if you pass away. This option is not common in most corporate pension plans.</td>
<td>For example, if you choose a 10-year period certain, you will receive payments for 10 years, and if you die during that period, your beneficiary will receive the balance of payments.</td>
</tr>
<tr>
<td>Lump Sum Distribution</td>
<td>Commonly determined from a formula using interest rate and life expectancy assumptions.</td>
<td>Lump sum distribution can (and should) be made in a tax-free rollover into an IRA account.</td>
</tr>
</tbody>
</table>
This paper extends the theory of general equilibrium in pure exchange economies to a prototype class of environments with private information.

and

examines again the role of securities in the optimal allocation of risk-bearing.

The first welfare theorem holds in this economy:
  - competitive equilibrium allocations are Pareto optimal.

The second fundamental welfare theorem however does not hold:
  - Not all Pareto optimal allocations can be supported as competitive equilibria.
Motivating Example

- at $T = 0$ all agents are the same
- at $T = 1$ fraction $\lambda(\theta)$ of agents receive a private shock $\theta \in \Theta = \theta_1, \theta_2$
- and their utility from consumption becomes $U(c, \theta)$
  - $U(c, \theta)$ is increasing, concave and continuously differentiable in $c$
  - $U'(\infty, \theta_1) = 0$ and $U(c, \theta_2) = \theta_2 c$, ($\theta_2 > 0$)
    - We only require type1 to be more risk averse than type2
- all agents receive endowment $e$ of consumption good with certainty and $U'(e, \theta_1) < \theta_2$. 

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Pareto Optimal Allocation

If $\theta$ were public, Pareto optimal allocation problem at $T = 0$ is:

$$\max_{c_1, c_2} \lambda(\theta_1)U(c_1, \theta_1) + \lambda(\theta_2) \times (\theta_2 c_2)$$

subject to $\lambda(\theta_1)c_1 + \lambda(\theta_2)c_2 \leq e$

- Pareto optimal allocation requires:
  $$U'(c_1^*, \theta_1) = \theta_2$$
  $$\lambda(\theta_1)c_1^* + \lambda(\theta_2)c_2^* = e$$
  i.e. marginal utilities are equated across states and the endowment is exhausted

- But with our assumptions, this requires $c_1^* < c_2^*$.
- If $\theta$ is private knowledge, we cannot implement this allocation since type 1 is always better off reporting her type is 2.

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Lotteries can solve this incentive compatibility problem.

Since type 2 is risk neutral, she is indifferent between:
- receiving $c_2^*$ with certainty
- receiving $c_3^*$ with probability $\alpha^* = c_2^*/c_3^*$ and consumption 0 with probability $1 - \alpha^*$.

But for $c_3^*$ sufficiently large, type 1 agents prefer consuming $c_1^*$ for sure instead of type 2 agents allocation.

Thus with lotteries we can achieve an allocation that is both Pareto optimal and incentive compatible.
Lotteries

• One can mimic the effects of a lottery by indexing on the basis of a naturally occurring random variable that is unrelated to preferences and technology, provided that the random variable has a continuous density.

• Agents are required to surrender their endowment $e$ to the broker and then, subsequent to the revelation of the shocks, they have a choice between two distribution centers.

• If they choose the first, they are guaranteed $c_1^*$ units of the good.

• If they choose the second, they receive $c_3^*$ units if it is available.

  • Households choosing the second center are imagined to arrive in a random fashion and to receive $c_3^*$ on a first-come, first-served basis.

• Agents are not permitted to recontract contingent upon whether or not they are served.
Imagine households can buy and sell contracts (make commitments) in a planning period \((T = 0)\) market.

Commitments can be conditional on households’ individual circumstances (i.e. their private shocks \(\theta\))

- of course, households will choose the option which is best given its individual circumstance.
- W.L.O.G. we can restrict to options such that household announce its individual shocks truthfully.

We allow options to affect random allocation of consumption good.
Contracts as a Bundle-I

Without Lotteries. Simplest notation is one good, but here to make sense need vector, otherwise no trade without lotteries.

- \( c(\theta) \) is the contract contingent on \( \theta \), \([c(\theta), \theta]\)
- Then \( U[c(\theta), \theta] \geq U[c(\theta'), \theta] \) for all \( \theta, \theta' \in \Theta \)
- The expected utility of contract \([c(\theta), \theta]\) for \( \theta \in \Theta \) is:

\[
W\{[c(\theta), \theta]\} = \sum_{\theta} \lambda(\theta) U[c(\theta), \theta]
\]

- Competitive Market
  - Households maximize in the standard problem by purchasing incentive compatible contracts \([c(\theta), \theta]\) \( \theta \in \Theta \), taking some pricing function \( p(\theta) \) \( \theta \in \Theta \) as given:

\[
\max \sum_{\theta} \lambda(\theta) U[c(\theta), \theta] \\
\text{s.t.} \quad \sum_{\theta} p(\theta) c(\theta) \leq \sum_{\theta} p(\theta) \varsigma
\]

- So it is as if selling endowment (\( \varsigma \)) and buying \( \theta \) contingent consumption back
- Equivalent with excess demand, or supply, for each \( \theta \), hence insurance

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Contracts as a Bundle-II

- A broker dealer offering contracts \([y(\theta), \theta \in \Theta]\), where \(y(\theta) > 0\): broker dealer is giving out to those who announce \(\theta\), indemnity, ex-post
- \(y(\theta) < 0\): broker dealer is taking in from those who announce \(\theta\), premium, ex-post
  - Revenue is \(\sum_\theta p(\theta)y(\theta)\)
  - Feasible trading set is defined by \(\sum_\theta \lambda(\theta)y(\theta) \leq 0\)
Formally an insurance contract can be shown by

\[ x(c, \theta), c \in C, \theta \in \Theta \]

- If household announce its shock \( \theta \), in the consumption period receive \( c \) with probability \( x(c, \theta) \).
  - of course \( 0 \leq x(c, \theta) \leq 1 \) and \( \sum_c x(c, \theta) = 1 \)
- Households buy these insurance contracts in the planning period market.
- Households endowments can be shown by probability measures \( \zeta(c, \theta), \theta \in \Theta \) each putting mass one on the endowment point \( e \).
- These endowments are sold in the planning period market.
Competitive Market

- In summary households maximize:

$$\max_{x(c, \theta)} \sum_{\theta} \lambda(\theta) \sum_{c} x(c, \theta) U(c, \theta)$$

s.t. $$\sum_{\theta} \sum_{c} p(c, \theta) x(c, \theta) \leq \sum_{\theta} \sum_{c} p(c, \theta) \zeta(c, \theta)$$

and incentive compatibility

- We also assume there are firms or intermediaries that make commitments to buy and sell the consumption good.
Competitive Market

- Firm production $y(c, \theta)$ delivers $c$ units of consumption if agent announce her type is $\theta$.
- Production set of each firm is defined by
  \[
  Y = \left\{ y(c, \theta), c \in C, \theta \in \Theta : \sum_{\theta} \lambda(\theta) \sum_{c} cy(c, \theta) \leq 0 \right\}
  \]
- (intermediary effectively facing aggregate resource constraint)
- This requires each firm not deliver more of the single consumption good in the consumption period than it takes in.
- $Y$ displays constant return to scale. So we can assume we only have one price taker firm.
- $y(c, \theta)$ is passive:
  - $y(c, \theta) > 0$: firm is giving away.
  - $y(c, \theta) < 0$: firm is taking in.
Competitive Market

- Firm problem is:
  \[
  \max \sum_{\theta} \sum_{c} p(c, \theta)y(c, \theta)
  \]

- Equilibrium price system \( p^*(c, \theta) \) must satisfy
  \[
  p^*(c, \theta) = \lambda(\theta)c
  \]

- This corresponds to actuarially fair insurance.
  - Price of A-D security which pays \( c \) at state \( \theta \) is just equal probability of the state \( \times \) consumption in that state
Welfare Theorems-I

- An allocation \((x_i)\) is implementable if it satisfies the resource constraints and a no-envy constraint

\[
W(x_i, i) \geq W(x_j, i) \quad \forall i, j
\]

- An allocation is a Pareto optimum if it is implementable and there does not exist an implementable allocation \((x'_i)\) such that

\[
W(x'_i, i) \geq W(x_i, i) \quad \text{with a strict inequality for some } i.
\]

Definition of Competitive Equilibrium
- A competitive equilibrium is
  - a state:\([(x^*_i), y^*]\)
  - a price system \(v^*\)
- such that:
  1. for every \(i\), \(x^*_i\) maximizes \(W(x_i, i)\) subject to \(x_i \in X\) and \(v^*(x_i) \leq v^*(\zeta)\)
  2. \(y^*\) maximizes \(v^*(y)\) subject to \(y \in Y\)
  3. \(\sum_{i=1}^{n} \lambda(i) x^*_i - y^* = \zeta\)
Welfare Theorems-II

First Welfare Theorem
If the allocation \([(x_i^*), y^*]\), together with the price system \(v^*\), is a competitive equilibrium and if no \(x_i^*\) is a local saturation point, then \([(x_i^*), y^*]\) is a Pareto optimum.

Second Welfare Theorem
With private information, there is no guarantee that every Pareto optimum can be supported by a quasi-competitive equilibrium with an appropriate redistribution of wealth.
It is true that a separating hyperplane exists such that \(y^*\) maximizes value subject to the technology constraint, but \(x_i^*\) does not necessarily minimize value over the set \(\{x_i \in X_i : W(x_i, i) \geq W(x_i^*, i)\}\). Rather, it minimizes value over the set \(\{x_i \in X_i : W(x_i, i) \geq W(x_i^*, i) \text{ and } W(x_i, j) \leq W(x_j^*, j) \text{ for } j \neq i\}\).
Need no envy condition.