Failure of the Welfare Theorems
14.04 Intermediate Micro Theory: Lecture 22

Robert M. Townsend

Fall 2019
Outline

- Failure of the Second Welfare Theorem - Nonconvexity
- Failure of the First Welfare Theorem - Local satiation
- Failure of the First Welfare Theorem- Pollution, Fixed with Markets in Rights
- Failure of the First Welfare Theorem - Externalities generally, rights for assignment to others
- Failure of First Welfare Theorem- Infinite Horizon and Infinite wealth
Take an economy $\mathcal{E} = \{ \{ X_i, u_i (\cdot), \omega_i \}_{i=1}^I, \{ Y_j \}_{j=1}^J, \{ \theta_{ij} \} \}$ such that:

1. $X_i \subseteq \mathbb{R}_+^L$ are convex, open sets, for all $i = 1, 2, \ldots I$
2. $Y_j \subseteq \mathbb{R}^L$ are convex and closed sets, and admit a concave transformation function $F_j : \mathbb{R}^L \to \mathbb{R}$; i.e.

$$Y_j = \{ y \in \mathbb{R}^L : F_j(y) \geq 0 \} \quad (13)$$

3. Preferences are given by concave and locally non-satiated utility functions $u_i : X_i \to \mathbb{R}$ (equivalent to preferences being rational, convex, continuous and locally non-satiated)

4. There exist $(\tilde{x}, \tilde{y})$ such that $\tilde{x} \in X_i$, $F_j(y_j) > 0$ for all $j = 1, \ldots, J$ and $\sum_i \tilde{x}_i \ll \bar{\omega} + \sum_j \tilde{y}_j$

**Theorem (Second Welfare Theorem)**

*Under assumptions 1 to 4, for any $\lambda \in \Delta^I$ there exist a price vector $p \in \mathbb{R}_+^L$ and a vector of wealth levels $w^* \in \mathbb{R}_+^I$ such that $(x^*, y^*, p, w^*)$ is a Walrasian equilibrium with transfers of $\mathcal{E}$*
Second Welfare Theorem for 1x1 model

(i) : Pareto Optimal Allocation

(ii) : 2nd Welfare Theorem

\[ V^* = \{(x_1, x_2) : u(x_1, x_2) \geq u(x^*)\} \]

\[ Y + \omega \]

\[ p^* = \]
Non-convexities

How can this construction fail then? Suppose that preferences were not convex, so that at $V^*$ were not convex. This may make it impossible to be able to separate both sets by a straight line (and hence, cannot be enforced as an equilibrium). The same thing happens when we have non-convex technology.

Figure: Non-convexity of preferences

Figure: Non-convexity of technology
Graphical Representation

**Figure 2.1.** Competitive equilibrium represented by a separation of sum sets.
First Welfare Theorem

Let \((x^*, y^*, p, w)\) be a price equilibrium with transfers. Then, if preferences are rational and locally non-satiated, then \((x^*, y^*)\) is a Pareto Optimal allocation.

Note that for this theorem, we need not assume anything about the consumption set \(X_i\) for each agent, other that it has to be consistent with the requirement of local non satiation (i.e. for any given bundle there must exist an arbitrarily close bundle that is strictly preferred).

We do not have to assume anything about the production sets either, other than the implicit assumption that \(Y_j \neq \emptyset\) for all \(j\) (because we know there exist some \(y^*\) that is part of a price equilibrium with transfers).
Local Non-satiation in the First Welfare Theorem
One agent that consumes 2 goods. A firm transforms good 1 into good 2, according to a production function $y_2 = F(-y_1)$. However, the firm generates pollution $P(y_2)$ per unit produced, that affects the agents utility. Preferences are given by $u = u(x_1, x_2, P)$.

Since there is only one consumer, Pareto Optimal allocations can be characterized by the solutions to the following programing problem:

$$\max_{(x_1, x_2, y_1, y_2)} u(x_1, x_2, P(y_2))$$  \hspace{1cm} (4)

subject to

$$x_1 = 1 + y_1 \text{ (multiplier } = \gamma_1)$$  \hspace{1cm} (5)

$$x_2 = y_2 \text{ (multiplier } = \gamma_2)$$  \hspace{1cm} (6)

$$y_2 \leq F(-y_1) \text{ (multiplier } = \gamma_3)$$  \hspace{1cm} (7)
Analytical Example II: Pollution

Which give us the following first order conditions:

\[
\frac{\partial L}{\partial x_1} = \frac{\partial u}{\partial x_1} - \gamma_1 = 0 \quad \frac{\partial L}{\partial x_2} = \frac{\partial u}{\partial x_2} - \gamma_2 = 0
\]

\[
\frac{\partial L}{\partial y_1} = -\gamma_3 \frac{\partial F}{\partial y_1} + \gamma_1 = 0 \quad \frac{\partial L}{\partial y_2} = \frac{\partial u}{\partial P} \frac{\partial P}{\partial y_2} + \gamma_2 - \gamma_3 = 0
\]

Simplifying we get the following FOC

\[
\frac{1}{dF/dy_1} = \frac{\partial u/\partial x_2 + (\partial u/\partial P)(\partial P/\partial y_2)}{\partial u/\partial x_1}
\]

So:

\[
MRS^{PO} (x_1^*, x_2^*) \equiv -\frac{dx_1}{dx_2} = \frac{\partial u/\partial x_2}{\partial u/\partial x_1} = \frac{1}{dF/dy_1} - \frac{1}{\partial u/\partial x_1} \left( \frac{\partial u}{\partial P} \frac{\partial P}{\partial y_2} \right)
\]  (8)

Condition 8 together with the resource constraints 5 and 6 characterizes the Pareto Optimal Allocation \((x_1^{PO}, x_2^{PO}), (y_1^{PO}, y_2^{PO})\)
Walrasian Equilibrium I

The first solution concept we introduce is Walrasian equilibrium for the economy with only 2 commodities \((x_1, x_2)\) and a single firm. The firm’s profit maximization problem is

\[
\max_{(y_1, y_2)} \pi \equiv p y_2 + y_1 \quad \text{s.t.} \quad y_2 \leq F(-y_1) \tag{9}
\]

Substituting \(y_2 = F(-y_1) \iff y_1 = -F^{-1}(y_2)\) into the profit function, we get the following first order condition with respect \(y_1\):

\[
\frac{\partial}{\partial y_2} \left[ p y_2 - F^{-1}(y_2) \right] = 0 \iff p = \frac{1}{\partial F / \partial y_1} \tag{10}
\]

using the inverse function theorem.
Walrasian Equilibrium II

On the other hand, the household solves its problem, taking pollution $P = P(y_2)$ and profits $\pi$ as given:

$$\max_{(x_1, x_2)} u(x_1, x_2, P). \text{ s.t. } x_1 + px_2 \leq 1 + \pi$$

which gives the first order condition:

$$MRS = p \iff \frac{\partial u/\partial x_2}{\partial u/\partial x_1} = p \quad (11)$$

Using 10 into 11 we get

$$MRS^e = \frac{1}{dF/dy_1} < \frac{1}{dF/dy_1} - \frac{1}{\partial u/\partial x_1} \left( \frac{\partial u}{\partial P} \frac{\partial P}{\partial y_2} \right) = MRS^{PO}$$

That is, $MRS$ in equilibrium is too small, we want $x_2$ to be smaller (too much of it is being produced) and hence $x_1$ bigger- use less as an input and eat more.
Walrasian Equilibrium with Pollution rights

Why does this happen? There exists an extra commodity, called “pollution” that has no market. In principle, the firm could purchase “pollution rights” from the consumer, making it a second input for production (and not only $y_1$).

Suppose that there is a market for “pollution permits”: a firm that wants to produce $y_2$ has to buy also the rights to create a pollution of $P = P(y_2)$. We endow the household with the rights to sell any level of permits it wants to firms, $\overline{P}$, at a fixed price. The household maximization problem is now:

$$\max_{(x_1, x_2, \overline{P})} u(x_1, x_2, \overline{P})$$

subject to

$$x_1 + px_2 \leq 1 + \pi + q\overline{P}$$

where $q$ is the unit price of the permit, per unit of pollution, and $\pi$ is the firm’s profits. Letting $\mu$ be the Lagrange multiplier of the budget constraint, the first order conditions are

$$\frac{\partial u}{\partial x_1} = \mu, \quad \frac{\partial u}{\partial x_2} = \mu p, \quad -\frac{\partial u}{\partial \overline{P}} = \mu q$$
Walrasian Equilibrium with Pollution Rights II

Therefore, in equilibrium:

\[ MRS = p \text{ and } -\frac{\partial u/\partial P}{\partial u/\partial x_1} = q \]  \hspace{1cm} (13)

On the firm side, now it has to buy the permits for the pollution it generates, which is an extra input in their production function:

\[
\max_{(y_1,y_2,-P) \in Y} py_2 + y_1 + q (-P) = \max_{y_2 \geq 0} py_2 - F^{-1}(y_2) - qP(y_2)
\]

So the first order condition is now

\[ p = \frac{1}{\partial F/\partial y_1} + q \frac{\partial P}{\partial y_2} \]  \hspace{1cm} (14)

In equilibrium, \( q \) is such that the market for pollution permits clear: \( \overline{P} = P \)

Putting together equations 13 and 14 we get

\[
MRS^{eq} = p = \frac{1}{\partial F/\partial y_1} + q \frac{\partial P}{\partial y_2} = \frac{1}{\partial F/\partial y_1} - \frac{1}{\partial u/\partial x_1} \left( \frac{\partial u}{\partial P} \frac{\partial P}{\partial y_2} \right)
\]

which coincides with the condition that defines the Pareto Optimal allocation, which makes this new competitive equilibrium, efficient.
Walrasian Equilibrium with Pollution Rights III

In the previous example, the consumer had the pollution rights which were sold to the firm. Alternatively, we could endow firms with pollution rights allow them to trade those rights (cap and trade).

Consider a setting with two firms, $a$ and $b$, endowed with pollution rights $P_a$ and $P_b$:

$$
\max_{y_{2j} \geq 0} py_{2j} - F_j^{-1}(y_{2j}) + q \left( P_j - P_j(y_{2j}) \right) \quad j = a, b
$$

(15)

Sell it all and buy back what is needed- not selling is the opportunity cost

So the first order condition is now

$$
p = \frac{1}{\partial F_j / \partial y_{1j}} + q \frac{\partial P}{\partial y_{2j}}
$$

(16)

If $\frac{\partial P_a}{\partial y_{2a}} > \frac{\partial P_b}{\partial y_{2b}}$, then the firm who pollutes more at the margin (firm $a$) will cut production compared to $b$, given the firm internalizes the cost of buying the pollution rights.
Walrasian Equilibrium with Pollution Rights IV

The idea behind the concept of “production externalities” (as well as “consumption externalities”) is that the actual choices of other agents affect the utilities of the household. This is violated by having a larger consumption than just $\mathbb{R}^L$, but rather $X_i \subseteq \mathbb{R}^L \times \mathbb{R}^{L(i-1)} \times \mathbb{R}^J$ (i.e. household also cares about the consumption bundles of all other households, and also cares about the production plans of all firms). However, markets only exist for the $L$ commodities that the consumer buys, so it takes the decisions of other agents as given.

In the example, the way to solve the problem is to let consumers sell pollution permits. In particular, we think of the consumer as having an endowment of permits, and selling them to firms. However, we can rewrite the budget constraint 12 as

$$x_1 + px_2 + q(-P) = 1 + \pi$$

so we can think of $(-P)$ as the commodity “absence of pollution” with price $q$, and that firms also produce (instead of just $y_2$).

Externalities can always be interpreted as a *missing market* problem.

Arrow (1969) illustrates the general point:

Consider a pure exchange economy. Let $x_{ik}$ be the amount of the $k$–th commodity consumed by the $i$–th individual ($i = 1, ..., n; k = 1, ..., m$) and $x_k$ be the amount of the $k$–th commodity available. Suppose in general that the utility of the $i$–th individual is a function of the consumption of all individuals (not all types of consumption for all individuals need actually enter into any given individual’s utility function); the utility of the $i$–th individual can be written $U_i(x_{i1}, ..., x_{inm})$. We have the obvious constraints.

\[ \sum_i x_{ik} \leq x_k \tag{17} \]

\[ x_{jik} = x_{ik}, \quad j = 1, ..., n \tag{18} \]

With this notation a Pareto-efficient allocation is a vector maximum of the utility functions $U_j(x_{j1}, ..., x_{jnm})$, subject to the constraints (17) and (18). Because of the notation used, the variables appearing in the utility function relating to the $j$–th individual are proper to him alone and appear in no one else’s utility function. If we understand now that there are $n^2m$ commodities ($n$ values for $i$ and for $j$, $m$ values for $k$), indexed by the triple subscript $jik$, then the Pareto-efficiency problem has a thoroughly classical form. There are $n^2m$ prices, $P_{jik}$, attached to the constraints (18), plus $m$ prices $q_k$, corresponding to constraints (17).

Following the maximization procedure formally, we see, much as in Samuelson [1954], that maximizing the lambda-weighted sum of utilities subject to constraints 17 and 18, Pareto efficiency is characterized by the conditions:

\[ \lambda_j \frac{\partial U_j}{\partial x_{jik}} = P_{jik} \]  
\[ \sum_j P_{jik} = q_k \]

where \( \lambda_j \) is the reciprocal of the marginal-utility of income for individual \( j \). (These statements ignore corner conditions; which can easily be supplied.)

Condition (20) can be given the following economic interpretation: Imagine each individual \( i \) to be a producer with \( m \) production processes, indexed by the pair \((i, k)\). Process \((i, k)\) has one input namely commodity \( k \), and \( n \) outputs, indexed by the triple \((j, i, k)\). In other words, what we ordinarily call individual \( i \)'s consumption is regarded as the production of joint outputs, one for each individual whose utility is affected by individual \( i \)'s consumption.

The point of this exercise is to show that by suitable and indeed not unnatural reinterpretation of the commodity space, externalities can be regarded as ordinary commodities, and all the formal theory of competitive equilibrium is valid, including its optimality.
Household $j$ as consumer

\[
\max_U U_j(x_{j1}, \ldots, x_{jn}) \\
\text{s.t. } \sum_i \sum_k P_{jik} x_{jik} = \sum_k q_k \omega_{jk}
\]

Taking FOC:

\[
\frac{\partial U_j}{\partial x_{jik}} = \mu P_{jik}
\]
Household $i$ as a firm is using as input $x_{ik}$ over goods $k$ that it is consuming, but now call it $-y_{ik}$, where input is negative, to produce $x_{jik}$ for other households $j$, $j = 1, 2, ... n$.

$$\max \sum_{j=1}^{m} (P_{jik}y_{jik} + q_ky_{ik})$$

s.t. $y_{jik} + y_{ik} = 0 \forall j \forall k$

Taking FOC:

$$P_{jik} = \theta_{jk}$$

$$q_k = \sum_j \theta_{jk}$$
Another failure of the first welfare theorem- infinite commodity spaces as in overlapping generations
Proof of FWT I

Using this lemma, we can now prove the First Welfare Theorem: that is, that any price equilibrium with transfers gives a Pareto Optimal allocation.

Proof.

Suppose, by contradiction, that there exists some other feasible allocation \((x, y)\) such that \(x_i \succeq_i x_i^*\) for all \(i\) and there exist \(i' : x_{i'} \succ_{i'} x_{i'}^*\). We must have that \(p x_{i'} > p x_{i'}^*\) and, using the previous Lemma, we must also have that \(p x_i \geq p x_i^*\). Therefore

\[
\sum_{i=1}^{l} px_i > \sum_{i=1}^{l} px_i^*
\] (1)
Proof of FWT II

Because \( y^*_i \) maximizes profits given prices, we must have that for all \( j \):
\[
py^*_j \geq py_j
\]
which implies that:
\[
\sum_{j=1}^{J} py^*_j \geq \sum_{j=1}^{J} py_j \tag{2}
\]

Putting together (1) and (2), and using the feasibility of \((x^*, y^*)\):
\[
\sum_{i=1}^{I} px_i > \sum_{i=1}^{I} px^*_i = p\bar{\omega} + \sum_{j=1}^{J} py^*_j \geq p\bar{\omega} + \sum_{j=1}^{J} py_j \implies \tag{3}
\]
\[
p \left( \sum_{i=1}^{I} x_i - \bar{\omega} - \sum_{j=1}^{J} y_j \right) > 0 \implies \sum_{i=1}^{I} x_i \neq \bar{\omega} + \sum_{j=1}^{J} y_j
\]

So \((x, y)\) was not feasible.
Households, How Many?

• Overlapping generations
  – Classic example
  – Time and number = infinity
AN EXACT CONSUMPTION-LOAN MODEL OF INTEREST WITH OR WITHOUT THE SOCIAL CONTRIVANCE OF MONEY*

PAUL A. SAMUELSON
Massachusetts Institute of Technology

My first published paper has come of age, and at a time when the subjects it dealt with have come back into fashion. It developed the equilibrium conditions for a rational consumer’s lifetime consumption-saving pattern, a problem more recently given by Harrod the useful name of “hump saving” but which Landry, Böhm-Bawerk, Fisher, and others had touched on long before my time.² It dealt only with a single individual and did not discuss the mutual determination by all individuals of the market interest rates which each man had to accept parametrically as given to him.

Now I should like to give a complete general equilibrium solution to the determination of the time-shape of interest rates. This sounds easy, but actually it is very hard, so hard that I shall have to make drastic simplifications in order to arrive at exact results. For while Böhm and Fisher have given us the essential insights into the pure theory of interest, neither they nor other writers seem to have grappled with the following tough problem: in order to define an equilibrium path of interest in a perfect capital market endowed with perfect certainty, you have to determine all interest rates between now and the end of time; every finite time period points beyond itself!

Some interesting mathematical boundary problems, a little like those in the modern theories of dynamic programming, result from this analysis. And the way is paved for a rigorous attack on a simple model involving money as a store of value and a medium of exchange. My essay concludes with some provocative

* Research aid from the Ford Foundation is gratefully acknowledged.


² As an undergraduate student of Paul Douglas at Chicago, I was struck by the fact that we might, from the marginal utility schedule of consumptions, deduce saving behavior exactly in the same way that we might deduce gambling behavior. Realizing that, watching the consumer’s gambling responses to varying odds, we could deduce his numerical marginal utilities, it occurred to me that, by watching the consumer’s saving responses to varying interest rates, we might similarly measure his marginal utilities, and thus the paper was born. (I knew and pointed out, p. 155, n. 2; p. 160, that such a cardinal measurement of utility hinged on a certain refutable "independence" hypothesis.)
Population of US by age and sex

See future lecture on inefficiency
Infinite Agents

Environments with infinite agents arise when considering *overlapping generation* models, as in Esteban, Mitra, and Ray (1994).

Consider an economy with \( I = \infty \) agents, two commodities: \( x_1 \) and \( x_2 \). Endowments are \((\omega_{1,i}, \omega_{2,i}) = (1, 1)\) and preferences given by \( u(x_1, x_2) = \ln(x_1) + \ln(x_2)\). Given prices \((p_1, p_2) = (1, p)\) the demand functions for all agents are

\[
x_1 = \frac{1}{2} (1 + p), \quad x_2 = \frac{1 + p}{2p}
\]

We will show that \( p = 1 \) is part of an equilibrium: see that if \( p = 1 \) then

\[
x_1 = x_2 = 1
\]

i.e. agents eat their own endowment, which means that the aggregate resource constraint needs to be satisfied, even when we restrict to any finite subset of agents. Is this allocation Pareto Optimal? The answer is: **NO**. Consider the following allocation:

\[
x = ((2, 2), (1, 1), (1, 1), ...)\]
Infinite Agents

Agent $i = 2$ gives her endowment to agent $i = 1$ (who now has $(2, 2)$), and agent $i = 3$ compensates her by giving her her own endowment, making $i = 2$ indifferent between this new allocation and the equilibrium allocation. To compensate agent $i = 3$ gives her endowment to $i = 2$ and so on.

Since there is an infinite number of agents, this allocation is feasible (which would not if we had a finite number of agents!). Since agent $i = 1$ is strictly better off, and $i \geq 2$ are indifferent (since they have exactly the same consumption bundle). Then, what went wrong with the First Welfare Theorem?

To understand this, we need to understand the most crucial part of the proof: the way we show that an alternative Pareto Dominating allocation cannot be feasible, is by showing that it cannot be purchased with the aggregate resources of the economy. This is expressed by condition 3

$$
\sum_i p x_i > \sum_i p \omega_i \implies \sum x_i \neq \sum_i \omega_i
$$

(22)
Now, condition 22 holds only when $\sum_i p\omega_i < \infty$. However, in the equilibrium allocation we consider:

$$\sum_i p\omega_i = \sum_i (1,1)^T (1,1) = \sum_i 2 = \infty$$

and hence condition 22 does not need to hold. Hence, the conclusion of the First Welfare Theorem does not hold. What would happen if, instead, at the equilibrium prices $p^*$ we have that the value of the endowment $\sum_i p^*\omega_i < \infty$? It turns out that the First Welfare theorem holds whenever the equilibrium prices are such that the value of the aggregate endowment is finite.

**Theorem (First Welfare Theorem - General version (Acemoglu 2010))**

Let $(x^*, y^*, p, w)$ be a Walrasian equilibrium with transfers, and suppose $I, L \in \mathbb{N} \cup \{\infty\}$. Then, if preferences are locally non-satiated and the value of the aggregate endowment is finite

$$p\bar{\omega} \equiv \sum_{i=1}^I \sum_{l=1}^L p_l \omega_{il} < \infty \quad (23)$$

Then $(x^*, y^*)$ is a Pareto Optimal allocation.
Proof.

Same as in the case with finite agents, since equation 23 then implies that if $p x_i \geq w_i$ for all $i$ and $p x_j > w_j$ for some $j$, then

$$\sum_i p x_i > \sum_i w_i = \sum_i p w_i$$

This does not need to hold when $\sum p w_i = \infty$, since we could have the inequality “$\infty > \infty$” which will not cause the contradiction in 3.