A Market-Based Solution for Fire Sales and Other Pecuniary Externalities*

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Abstract

Pecuniary externalities are removed with market exchanges that internalize agent types' influence on future prices. Agents choose in the initial period from among various possible prices they want to prevail in the future and buy or sell rights in these market exchanges for future trade. One can think of contemporary exchanges or trading houses, earmarked by these specified future prices. Each agent can choose the exchange it wants without regard to what any other agent is doing. But crucially, the right to trade in each and every exchange is priced. The fee structure has a per unit price and quantity decomposition: a price, as determined by the exchange chosen, times the quantity of type-specific future excess demand, as determined by its chosen future pre-trade position, namely, savings and security holdings, and the chosen future price.

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1 Introduction

Both developed and emerging economies have experienced episodes of rapid credit expansion followed, in some cases, by a financial crisis, with a collapse in asset prices, credit, and investment. There is also a literature on fire sales in financial markets, e.g., Gorton and Metrick (2012). However, as Lorenzoni (2008) emphasizes, if the private sector had accurate expectations and correctly incorporated risk in its maximizing decisions, yet still decided to borrow heavily during booms, it means that the expected gain from increased investment more than compensated for the expected costs of financial distress. Thus one needs to understand how, and under what conditions, this private calculation leads to inefficient decisions at the social level. What is the externality? Likewise, how can it be remedied? Do we need regulation and government intervention, or can innovative market structure that internalizes the externality solve the problem.

There is a literature in the wake of the U.S. financial crisis that has focused on pecuniary externalities as the source of the problem. This literature seeks policy interventions and regulations to remedy the associated distortions, e.g., balance sheet effects, amplifiers and fire sales. Under pecuniary externalities, trading on a market adversely affects others via the revaluation of traded items. Solutions range from regulation of portfolios, restrictions on saving or credit, interest rate restrictions, fiscal policy, or taxes and subsidies levied by the government. However, general equilibrium theory suggests in other contexts that bundling, exclusivity and suitably designed additional markets for the objects associated with externalities could internalize those externalities, without the need of further policy interventions, or the need to quantify interventions, as the latter requires yet more information. Ex ante competition and equilibrium with market-determined prices for rights to trade in these additional markets can achieve a constrained-efficient allocation. Here we follow general equilibrium theory and remove pecuniary externalities in this way.

To reiterate, our solution to the problem of pecuniary externalities is a market-based solution, creating market infrastructure, rights to trade, and prices of those rights in such a way that a constrained-efficient allocations can be decentralized with a price system. The two fundamental welfare theorems hold. There is no need for the government to impose quotas, allocate rights or calculate marginal taxes, as arguably these require considerable information and nontrivial calculation. It is beyond the scope of this paper to repeat the arguments of Smith and Hayek, the Lange Lerner debates, and the work of Hurwicz, for example, but we do come back to the determination of prices in a section with the planner transformed into market maker and in the conclusion, with citations to the computation of equilibria and explicit market making mechanisms.

The influence of prices which can cause inefficiencies is akin to pollution, which has a remedy in competitive markets for the rights to pollute. We will draw an analogy between pollution and price externalities to explain what we do. Specifically, for pollution, consider an initial economy with two goods, one period, one representative price-taking consumer and one representative price-taking firm. The consumer is endowed with one good which can be consumed or used by firms to produce the second good which the household also values. However, that production comes with air or water pollution, which gives the household disutility. The competitive equilibrium in which this pollution is not priced is not at a social optimum; marginal rates of substitution in consumption and production do not line up, as they would in the planner's problem. But now suppose we create markets in the rights to pollute. Factories have to buy rights to emit pollution, a cost which lowers their profit. They choose how much to produce and how much to pollute, consistent with permits purchased. Households sell rights to suffer pollution, a revenue added to their budget, and choose how much pollution they want and how much to consume of the two goods. In the new decentralized market equilibrium, the demand for rights to pollute by firms and the supply of rights to suffer pollution by households will be equated by the appropriate price of rights, and money in equal amounts changes hands. The new equilibrium is Pareto optimal; it has some but less pollution. Of course, there needs to be some enforcement. Firms cannot pollute beyond rights purchased, as in cap and trade. The difference between cap and trade and the full market solution described here is that the quantity of permits is market determined and not fixed by the government.

Now consider an analogue economy with two goods, two periods, and two representative price-taking households, types a and b. There is no uncertainty. The decisions are intertemporal decisions, over the two time periods, and within-period decisions, across the two goods. Further, suppose that if there were no obstacles to trade and if markets were complete, then the environment is such that, in a competitive equilibrium, type a would be a lender and type b a borrower. However, suppose in contrast that borrowing cannot happen in equilibrium.[1] Further, only one of the two goods can be stored, good z. The relatively rich type a ends up smoothing consumption over time on its own, not by lending to type b but by saving good z. As a result, the price of the storage good z is low in the second period, as type a sells good z in the spot market then. This relative price is moving with saving, but both types take this equilibrium price as given. This relative price is the source of a pecuniary externality.

The stored good in this intertemporal example is like the input good in the initial pollution example. As with the pollution example where we needed markets for rights to pollute, here with this pecuniary externality we need markets for rights to trade at the future relative price. Agents of each type choose in the first period the relative price at which they want to trade in the second period. That is, agents choose in the first period one price from among various possible prices they want to prevail in the future. One can think of a first period exchange or trading house earmarked by each possible such relative price, p. Subsequently, we refer to these first period exchanges as price exchanges or for brevity, p-exchanges. Each agent can choose the price exchange it wants without regard to what any other agent is doing. But crucially, the right to trade in each and every p-exchange is priced. The fee structure has a per unit price and quantity decomposition, a price times type-specific excess demand, as determined by the relative price p of the price exchange chosen and previously chosen saving. The fees to engage and trade rights in these price exchanges are the market-determined decentralizing feature.

Our analysis extends well beyond the example, which is intended to be illustrative. In

[1] This can be derived from limited commitment that the type b borrowers' promise to pay must be backed by collateral held in storage. By assumption the would-be borrower type b has little of the collateral good, and so the competitive equilibrium with limited commitment has no borrowing and lending.

our more general setup any agent type can make a promise to deliver in the future, but all promises must be backed by sufficient collateral so that promises can be honored. We can allow uncertainty about future states of the world, and promises can be state-contingent, as would be the collateral constraints, holding state by state to ensure state-contingent promises are honored. We can also allow exogenously incomplete markets. With incomplete security markets we can drop the collateral requirement, but generically competitive equilibria are inefficient due to pecuniary externalities when there are multiple goods in spot markets. Trades in securities markets determine the distribution of income in spot markets, but by definition, when markets are incomplete, there is no way to hedge the resulting income movements across states (e.g., Geanakoplos and Polemarchakis, 1986; Greenwald and Stiglitz, 1986). In this case, as security positions move relative prices in multiple future states, rights are naturally a vector of rights over future states. We can remedy the pecuniary externality, so that allocations are constrained-efficient, though still not complete. Our solution is not about completing markets but remedying price externalities. Other environments include a fire sale economy (Lorenzoni, 2008) as per our introduction to this paper, and a liquidity-constrained economy (Hart and Zingales, 2013), where there is too much saving. We also extend our method to environments with information imperfections, namely a moral hazard contract economy with multiple goods and retrade in spot markets (e.g., Acemoglu and Simsek, 2012; Kilenthong and Townsend, 2011) and a Diamond-Dybvig economy where an agent's excess demands in interim bond markets is not known ex ante as each agent is subject to unobserved preference shocks that determine the direction of trade (e.g., Diamond and Dybvig, 1983; Jacklin, 1987). Our solution works in general as price is a "sufficient statistic" for the source of the problem, regardless of the underlying environment. This is after all the nature of pecuniary externalities.

Our contribution is related to Coase (1960) in its emphasis on rights. Our pollution example is one of his lead examples. However, the Coase theorem is about how any given initial arbitrary distribution of rights would not matter if there were bargaining and no trading frictions, just as the initial allocation of rights to pollute in cap and trade would not matter, as efficiency works through opportunity costs.[2] In contrast, for us, rights are

[2] We can relate our solution to Lindahl (1958) who uses agent-specific prices to solve a public goods

market determined. Thus, we are working not in the tradition of Coase (1960) but rather in the tradition of Arrow (1969), following Meade (1952), on the equivalence of solutions to planning problems and competitive equilibria with rights to trade in the objects causing non-pecuniary externalities. Keys are additional markets and excludability.[3] Of course we focus on pecuniary externalities.

The remainder of the paper proceeds as follows. Section 2 presents the saving economy to illustrate the ingredients. This includes Section 2.1, the competitive economy (with the externality due to a binding non-negativity constraint and the fact that agents taken prices as given). Section 2.2 presents the basic planner problem, making clear that the planner can take into account the equilibrium pricing function in the second period spot markets, the mapping from savings to price and how this impacts agent type value functions. Section 2.3 presents an equivalent market maker problem, introducing the language of rights and letting price be the planner control variable, which dictates saving, through the inverse pricing function. This provides a transition to the decentralized markets with trading rights in Section 2.4. A general economy is described and the welfare theorems and existence theorem are stated in Section 3. Section 4 concludes with some comments on implementation. Appendix A presents the proof of the second welfare theorem, and additional results are in the online Appendices.

2 A Saving Economy Illustrative of the Key Ingredients

This section features in notation the example economy of the introduction, a saving economy with no uncertainty. There are two periods, $t = 1, 2$. Planning takes place at the initial date,

problem when there is heterogeneity in willingness to pay. Though the per unit price of an exchange is common, type specific excess demands make the total fee agent specific.

[3]See Chapter 11 of Mas-Colell et al. (1995) for more about this distinction between Coase (1960) and Arrow (1969). Interestingly, Arrow (1969) is less concerned about excludability, an intrinsic part of creating the necessary markets, as he feels this has a natural counterpart in many real world problems. Arrow (1969) is more concerned about the obvious small numbers problem. However, this part is easy to remedy with a continuum of traders and positive mass of each trader type, as we do here.

hence $t = 1$ though there are spot markets and saving later in that period, as well. With the second date, $t = 2$, and no uncertainty; this is a pure intertemporal economy, making the point that the problem and its remedy has nothing to do with uncertainty. In particular, our rights are not trades in financial options.

There is a continuum of agents of measure one. The agents are however divided into two heterogeneous types $h = a, b$. Each type h consists of α^h fraction of the population. There are two consumption goods, which can be traded and consumed in each period t. Each unit of good z stored will become R units of good z at date $t = 2$.[4] Good w cannot be stored (is completely perishable). Let $k^h \in \mathbb{R}_+$ denote the saving (equivalent to the holding of good z) of an agent type h at the end of period $t = 1$ to be carried to period $t = 2$. Let good w be the numeraire good in each and every date. The price in terms of the numeraire at which the good z can be bought and sold in spot markets in the second period, at $t = 2$, is the key object associated with the pecuniary externality. The contemporary preferences of agent type h are represented by the utility function $u^h\left(c_w^h, c_z^h\right)$, which is continuous, strictly concave, strictly increasing in both consumption goods, and satisfies the usual Inada conditions. Each agent type h is endowed with good w and good z, $\mathbf{e}_t^h = \left(e_{wt}^h, e_{zt}^h\right) \in \mathbb{R}_+^2$ in period $t = 1, 2$.

For the numerical example we shall suppose each of the two types has an identical constant relative risk aversion (CRRA) utility function $u^h(c_w, c_z) = -\frac{1}{c_w} - \frac{1}{c_z}$, $h = a, b$. The endowment profiles are such that an agent type a is well endowed with 3 units of both goods in period $t = 1$ relative to one unit of both at $t = 2$, and vice versa for type b. Each type h consists of $\frac{1}{2}$ fraction of the population, i.e., $\alpha^h = \frac{1}{2}$. In a full unconstrained optimum, with $R = 1$, each agent would consume 2 units of each good in each period.

2.1 Competitive Equilibrium with Externalities

Definition 1. A competitive equilibrium is a specification of prices of good z in period $t = 1$ and $t = 2$, p_1 and p_2, respectively; consumptions $\left(c_{w1}^h, c_{z1}^h\right)$ at $t = 1$, saving k^h decision made at $t = 1$, and trades $\left(\tau_{w2}^h, \tau_{z2}^h\right)$ at $t = 2$ for each type h such that (i) agent type h solves

$$\max_{c_{w1}^h, c_{z1}^h, k^h, \tau_{w2}^h, \tau_{z2}^h} u^h\left(c_{w1}^h, c_{z1}^h\right) + u^h\left(e_{w2}^h + \tau_{w2}^h, e_{z2}^h + Rk^h + \tau_{z2}^h\right) \tag{1}$$

[4]Though R denotes the return on saving, in particular in all examples we set $R = 1$.

subject to the budget constraint in period $t = 1$, the budget constraint in period $t = 2$, and the non-negative saving constraint, respectively,

$$c_{w1}^h + p_1 \left(c_{z1}^h + k^h\right) = e_{w1}^h + p_1 e_{z1}^h, \tag{2}$$

$$\tau_{w2}^h + p_2 \tau_{z2}^h = 0, \tag{3}$$

$$k^h \geq 0, \tag{4}$$

(ii) the markets for good w and good z at $t = 1$, and trades at $t = 2$, respectively, clear

$$\sum_h \alpha^h c_{w1}^h = \sum_h \alpha^h e_{w1}^h, \tag{5}$$

$$\sum_h \alpha^h \left[c_{z1}^h + k^h\right] = \sum_h \alpha^h e_{z1}^h, \tag{6}$$

$$\sum_h \alpha^h \tau_{\ell 2}^h = 0, \forall \ell = w, z. \tag{7}$$

Necessary conditions for competitive equilibrium related to saving k^h are

$$p_1 = \frac{u_{z1}^h}{u_{w1}^h} = \frac{u_{z2}^h}{u_{w1}^h} R + \frac{\eta^h}{u_{w1}^h}, \forall h = a, b. \tag{8}$$

where $u_{\ell t}^h \equiv \frac{\partial u^h \left(c_{wt}^h, c_{zt}^h\right)}{\partial c_{\ell t}}$ for $\ell = w, z; t = 1, 2$, and η^h is the Lagrange multiplier for the non-negative saving constraint for an agent type h. The price of good z at $t = 1$, p_1, is derived from two components: (1) the value from the return R in the next period, and (2) the value from the fact that constrained agents would like to borrow but cannot, as saving cannot be negative.

To be more explicit about the spot market with price p_2 at $t = 2$, we present a dynamic programming formulation. An agent type h with given, predetermined saving k^h is free to choose at $t = 2$ spot trades $\left(\tau_{w2}^h, \tau_{z2}^h\right)$ to solve the following utility maximization

$$V^h \left(k^h, p_2\right) = \max_{\tau_{w2}^h, \tau_{z2}^h} u^h \left(e_{w2}^h + \tau_{w2}^h, e_{z2}^h + R k^h + \tau_{z2}^h\right) \tag{9}$$

subject to the spot market budget constraint (3) at $t = 2$. Of course, to be consistent, the excess demands for good w at $t = 2$, the $\tau_{w2}^{h*}\left(k^h, p_2\right)$ over types h, must satisfy the market-clearing conditions for trades (7) at $t = 2$.[5] In fact, the spot market equilibrium price p_2

[5] With Walras' law, excess demands for good z are implied.

can be defined as the price which makes (7) be satisfied, and this is a function of savings of both types, possibly zero (but here keeping endowments implicit): $p_2 = p_2\left(k^a, k^b\right)$. This function applies at $t = 2$ for all possible specifications of agent savings. The pricing function $p_2 = p_2\left(k^a, k^b\right)$ can be thought of as an equilibrium consistency constraint for prices and savings, with prices not only for actual savings but also any counter-factual savings. One then works backwards to the first date, to determine actual savings, in the competitive equilibrium. Of course, agents take prices as given, not the function. They do not need to think through that price is a function of savings and, with no mass for a trader of a given type, traders have zero influence, anyway.

Table 1: Equilibrium allocations with externalities.

	k^h	c^h_{w1}	c^h_{z1}	c^h_{w2}	c^h_{z2}	$U^h\left(\mathbf{c}^h\right)$
$h = a$	1.36	2.69	1.78	1.33	1.78	-2.2527
$h = b$	0	1.31	0.87	2.67	3.58	-2.5724

For the numerical example, we summarize the equilibrium allocation in Table 1 featuring saving k^h and consumption $c^h_{\ell t}$. Note that the first-best allocation features no saving, $k^a = k^b = 0$, and non-time-varying prices of good z, $p_1 = p_2 = 1$. The first-best allocation suggests that agent b would like to move resources backwards in time from $t = 2$ to $t = 1$, i.e., borrow and therefore will be constrained. The equilibrium with the externality present will have agent type b borrowing nothing and only trading in spot markets. This can be derived as endogenous as in footnote 1 or interpreted as exogenous incomplete markets. Agent type a will be saving on its own to smooth consumption over time. With the externality (denoted "ex"), the price of good z in period $t = 1$ is $p_1^{ex} = \left(\frac{4}{4-k^{ex}}\right)^2 = 2.2948$, and at date 2 is $p_2^{ex} = 0.5570$. Note that the price of good z is high at $t = 1$ relative to the first best since relatively much is put into storage, and likewise the price of good z is low at $t = 2$ when the storage is sold on the market.

2.2 The Planner Problem

The planner will choose the distribution of savings and first-period consumptions to solve the following programming problem:

$$\max_{\left(c_{w1}^h, c_{z1}^h, k^h\right)_h} \sum_h \lambda^h \alpha^h \left[u^h\left(c_{w1}^h, c_{z1}^h\right) + V^h\left(k^h, p_2\left(k^a, k^b\right)\right)\right] \tag{10}$$

subject to non-negative constraints on saving (4), and the resource constraints for good w and good z at $t = 1$, (5) and (6), respectively, as we now explain in more detail. Note that the value function V is already defined in the agent maximization problem (9).

The planner will maximize a λ-weighted sum of discounted type utilities subject to constraints: the resources constraints in the first period, that aggregate consumption cannot exceed aggregate endowments minus aggregate saving; the non-negativity constraint on savings $k^h \geq 0$ for each type (incomplete markets, no borrowing); and respecting that allocations will be determined in spot markets with endogenous market clearing prices $p_2 = p_2\left(k^a, k^b\right)$ (the planner is aware that savings influences the relative price, that agents in contrast do not take into account, hence the pecuniary externality in the decentralized competitive equilibrium and the analogy to pollution). As the λ weights are varied, this constrained maximum problem will trace out the frontier of the set of constrained-optimal allocations. Constrained optimal allocations take into account two key features. First, the planner cannot have more objects under its control than the agents do, i.e. the planner cannot impose a solution with borrowing at $t = 1$ to be repaid at $t = 2$ (that is, the planner cannot undo the incomplete markets). The planner can assign $t = 1$ consumption and savings, as even with incomplete markets agents do choose these. Second, in contrast, the planner cannot assign consumptions at $t = 2$. The control variables are transfers, not consumption allocations. Further these transfers must be the type specific excess demands of the agents at equilibrium prices p_2. An agent type h with saving k^h is free to choose even in the implementation of the planner problem spot trades $\left(\tau_{w2}^h, \tau_{z2}^h\right)$ to solve utility maximization of (9) subject to (3), yielding the value function $V^h\left(k^h, p_2\right)$. The planner has to take the existence, trade, and equilbrium in spot markets at $t = 2$ as a given.

But the planner does take into account that the equilibrium relative price p_2 will be determined by assigned savings at $t = 1$. More formally, let $\boldsymbol{k} = \left(k^h\right)_h$ denote the distribution of savings across agent types h, and let the price be $p_2 = g(\boldsymbol{k}, \boldsymbol{e})$, the same function as $p_2 = p_2\left(k^a, k^b\right)$ but now with the function g making endowments explicit. For example, with homogeneous homothetic preferences, the mapping can be written as a function of the aggregate ratio between the two goods, i.e., $g(\boldsymbol{k}, \boldsymbol{e}) = g\left(\frac{\sum_h \alpha^h e^h_{w2}}{\sum_h \alpha^h \left[Rk^h + e^h_{z2}\right]}\right)$. In this example, we can show that the equilibrium spot price is decreasing in individual type saving $k^{\tilde{h}}$, i.e., $\frac{\partial p_2\left(k^a, k^b\right)}{\partial k^{\tilde{h}}} = -\frac{\alpha^{\tilde{h}} R \sum_h \alpha^h e^h_{w2}}{\left(\sum_h \alpha^h \left[Rk^h + e^h_{z2}\right]\right)^2} g'\left(\frac{\sum_h \alpha^h e^h_{w2}}{\sum_h \alpha^h \left[Rk^h + e^h_{z2}\right]}\right) < 0$ as the function g is increasing in the aggregate ratio.

The necessary conditions[6] for constrained optimality are given by

$$\frac{u^h_{z1}}{u^h_{w1}} = \frac{u^h_{z2}}{u^h_{w1}} R + \frac{\mu^h}{\lambda^h \alpha^h u^h_{w1}} + \frac{1}{\lambda^h u^h_{w1}} \sum_{\tilde{h}} \lambda^{\tilde{h}} \alpha^{\tilde{h}} \frac{\partial V^{\tilde{h}}}{\partial p_2} \frac{\partial p_2}{\partial k^{\tilde{h}}}, \forall h = a, b. \tag{11}$$

where μ^h are Lagrange multipliers for the non-negative saving constraints. Note that the solutions to the planner problem vary with Pareto weights λ^h, tracing out all possible Pareto optimal allocations. Intuitively, the value or shadow price of good z consist of three components: The first two components were there also in the competitive equilibrium but the third here is new. It is the value from its effect on the future $t = 2$ price through the impact of savings on the equilibrium price function $\frac{\partial p_2\left(k^1, k^2\right)}{\partial k^{\tilde{h}}}$ for all agent types, quite naturally taken into account by the planner. As formally proved in Kilenthong and Townsend (2014), the optimal level of saving is lower than the equilibrium level. This is also the correct point to review the nature of the problem of pecuniary externalities.

Definition 2. A pecuniary externality arises when the equilibrium pricing function has a non-zero derivative with respect to savings $\boldsymbol{k}$ and the solutions to the planner problem (10) are different from the solutions (8) to the individual maximization problem.

The writing of the planners problem with value functions does not per se mean that there

[6]Given that the constraint set is not convex, the optimality conditions are necessary but may not be sufficient. This does not cause any problem to our externality argument here, as for that part we simply need to show that an equilibrium cannot be constrained optimal, i.e., does not satisfy the necessary optimal conditions (11). We overcome the non-convexity problem using a mixture representation as in Appendix A, where first-order conditions are necessary and sufficient. Finally the necessary condition (22) drops the common market assignment (14). More on that below.

are pecuniary externalities to be corrected.[7] What does generate a pecuniary externality in the competitive decentralization are genuinely incomplete security markets with multiple goods and saving, coupled with trading in spot market, as in Geanakoplos and Polemarchakis (1986). Indeed, the example of Greenwald and Stiglitz (1986) is close to our baseline savings-only example here; we establish in online Appendix E that the competitive decentralization (without rights) is not achieving a constrained optimum, as prices are moving with saving, and in Kilenthong and Townsend (2014) that there is too much saving (compare Table 1 with Table 2 below).

2.3 Market Maker Problem: A Transformed, Equivalent Planner

We can easily transform the problem to a fully equivalent one in which the planner is a market maker choosing price p_2 and then finding savings $\boldsymbol{k}$ to support that price with all possible prices considered. For example, with homogeneous homothetic preferences, $R = 1$, and agent type a doing all the savings, given any price p_2, we can use the inverse function $g^{-1}(p_2)$ taking $\boldsymbol{e}_2^h$ as given to go from price p_2 to savings k^a,

$$k^a = \frac{\sum_h \alpha^h e_{w2}^h}{g^{-1}(p_2)} - \sum_h \alpha^h e_{z2}^h \tag{12}$$

If both types could save and R were unrestricted, then $\sum_h \alpha^h R k^h$ would be on the LHS of (12).[8]

Notationally, let $\delta(p_2)$ be a binary $0, 1$ indicator for the planner's choice of p_2 as a market maker. The consumptions $\boldsymbol{c}_1^h(p_2) \equiv \left(c_{w1}^h(p_2), c_{z1}^h(p_2)\right)$ and savings $k^h(p_2)$ at $t = 1$ for type h are chosen also, respectively, by the planner as market maker, while spot trades $\boldsymbol{\tau}^h(p_2) \equiv \left(\tau_{w2}^h(p_2), \tau_{z2}^h(p_2)\right)$ at $t = 2$, are as derived earlier in (9) under individual type h maximization at p_2. Further, now define rights $\Delta^h\left(k^h(p_2), p_2\right) \equiv \tau_{w2}^{h*}\left(k^h(p_2), p_2\right)$ where the right hand side is the type h maximizing choice, the same as before. The planner is in full control of and assigns these rights to trade in the

[7] For example, in the original Arrow (1964) dynamic decentralization of the full optimum, a security paying off the numeraire is the control object chosen at $t = 1$, knowing that there will be retrading at prices at $t = 2$. A hybrid, step-wise planner problem would decentralize at $t = 2$ and retain the original planner problem at $t = 1$, as we do.

[8] This exploits the monotonicity and uniqueness of p_2 in the homothetic case, but, more generally, let the planner choose among p_2's implied by a given distribution of k^h's.

spot market $t=2$ at price p_2, but these rights must be consistent with the trades agent type h would want to do voluntarily at that time and price. The rights notation appears as redundant at this point as we already have the τ^h's, but this is an important if subtle step in the reformulation of the planner problem to the market markers problem. Rewriting clearing condition (7) for good $\ell = w$,

$$\delta(p_2) \sum_h \alpha^h \Delta^h \left(k^h(p_2), p_2\right) = 0, \forall p_2. \tag{13}$$

Constraints (13) hold for all possible choices of p_2, that is, for active and inactive markets, just as $p_2 = g(\boldsymbol{k}, \boldsymbol{e})$ held for any actual and counterfactual choice of k^h's. That is, the constraint is clearly operation $\delta(p_2) = 1$, but the market maker is also taking into account that there would have to be a market determined spot price at $t=2$ if any other p_2 were chosen instead. Finally, as both agents are assigned to the same, common market at price p_2, define individual type h assignment of the market maker to respect that:

$$\delta^h(p_2) \equiv \delta(p_2), \forall h = a, b, \tag{14}$$

so that (13) can be rewritten as

$$\sum_h \delta^h(p_2) \alpha^h \Delta^h \left(k^h(p_2), p_2\right) = 0, \forall p_2. \tag{15}$$

The equivalent transformed planner problem with Pareto weights $\left[\lambda^h\right]_h$ is defined as follows.

$$\max_{\left[\mathbf{x}^h\right]_h} \sum_{h,p_2} \lambda^h \alpha^h \delta^h(p_2) \left[u^h\left(c_{w1}^h(p_2), c_{z1}^h(p_2)\right) + u^h\left(e_{w2}^h + \tau_{w2}^h(p_2), e_{z2}^h + Rk^h(p_2) + \tau_{z2}^h(p_2)\right)\right] \tag{16}$$

where $\mathbf{x}^h = \left[\mathbf{c}_1^h(p_2), k^h(p_2), \boldsymbol{\tau}^h(p_2), \delta^h(p_2), \Delta^h\left(k^h(p_2), p_2\right)\right]_{p_2}$ denote a typical bundle or allocation assigned to an agent type h running over all p_2, subject to non-negative constraints on saving (17), the resource constraints for good w and good z at $t=1$ (18)-(19), the resource constraints for spot trades (20), the spot market budget constraints (21),

$$\delta^h(p_2) k^h(p_2) \geq 0, \forall h; p_2, \tag{17}$$

$$\sum_{p_2} \sum_h \delta^h(p_2) \alpha^h c_{w1}^h(p_2) = \sum_h \alpha^h e_{w1}^h, \tag{18}$$

$$\sum_{p_2} \sum_h \delta^h(p_2) \alpha^h \left[c_{z1}^h(p_2) + k^h(p_2)\right] = \sum_h \alpha^h e_{z1}^h, \tag{19}$$

$$\sum_h \delta^h(p_2)\,\alpha^h \tau^h_{\ell 2}(p_2) = 0, \forall \ell = w, z; p_2, \tag{20}$$

$$\delta^h(p_2)\left[\tau^h_{w2}(p_2) + p_2 \tau^h_{z2}(p_2)\right] = 0, \forall h; p_2, \tag{21}$$

the consistency constraints (13) substituting in the common market assignment across types (14) and hence (15).

The necessary conditions for constrained optimality that are comparable to the one of a competitive equilibrium (8) are given by

$$p_1 = \frac{u^h_{z1}}{u^h_{w1}} = \frac{u^h_{z2}}{u^h_{w1}} R + \frac{\mu^h(p_2)}{\lambda^h \alpha^h u^h_{w1}} - \frac{\mu_\Delta(p_2)}{\mu_{w1}} \Delta^h_k\left(k^h(p_2), p_2\right), \forall h = a, b. \tag{22}$$

where the derivative of rights $\Delta^h_k\left(k^h, p_2\right) \equiv \frac{\partial \Delta^h\left(k^h, p_2\right)}{\partial k}$, μ_{w1} are Lagrange multipliers for the resource constraints for good w in period $t = 1$, $\mu^h(p_2)$ are Lagrange multipliers for the non-negative saving constraints, and $\mu_\Delta(p_2)$ are the key Lagrange multipliers for the consistency constraints (15).[9]

The numerical example is displayed in Table 2 where, for the constrained optimal allocation, $p_2^{op} = 0.5974$ is higher than in the competitive equilibrium with externalities in Table 1 as there is less saving.[10]

Table 2: Constrained Optimal Allocation with Pareto weights $\lambda^1 = 0.778$ and $\lambda^2 = 0.222$.

	k^h	c^h_{w1}	c^h_{z1}	c^h_{w2}	c^h_{z2}	$\Delta^h\left(p_2^{op}\right)$	$U^h\left(\mathbf{c}^h\right)$
$h = a$	1.18	2.61	1.84	1.30	1.68	0.30	-2.2934
$h = b$	0	1.39	0.98	2.70	3.50	-0.30	-2.3904

The consistency constraints (13) in the transformed problem are binding even for $\delta(p_2) = 0$ in the optimal assignment of p_2. Intuition is provided by the original planner's problem where the planner must take into account that excess demands $\tau^{h*}_{w2}\left(k^h(p_2), p_2\right)$ with weight α^h must sum to zero for any choice of k^h's, that is, not simply the one chosen but for any counterfactual choice of k^h's. Consideration of all possibilities is what dictated the maximizing choice of a given p_2 to begin with.

To provide some intuition via a numerical example, and to anticipate the mixture notation in Appendix A, let $x^h(\mathbf{c}_1, k, \boldsymbol{\tau}, p_2, \Delta)$ denote the fraction of agents type h assigned the allocation of

[9]This optimal condition was derived by taking derivatives with respect to prices and then exploiting first order conditions as in Prescott and Townsend (1984b).

[10]The Pareto weights are chosen so that the optimal allocation corresponds to the competitive equilibrium with rights to trade, defined below, without transfers.

consumption $\mathbf{c}_1$ and saving k at $t = 1$, transfers $\boldsymbol{\tau}$ and being in p_2-exchange with rights Δ at $t = 2$, subject to a constraint analogous to (15), namely, summing over all objects other than p_2 gives the following:

$$\sum_h \alpha^h \sum_{\mathbf{c}_1, k, \boldsymbol{\tau}, \Delta} x^h \left(\mathbf{c}_1, k, \boldsymbol{\tau}, p_2, \Delta\right) \Delta = 0 \tag{23}$$

Let us give some slack to these consistency constraints of approximately 0.01 at two prices $p_2 = 0.6181$ (an inactive one, not chosen) and $p_2 = 0.5974$ (the active one, chosen). Reassign some fraction of agent type a, $x^a \left(\mathbf{c}_1, k, \boldsymbol{\tau}, p_2, \Delta\right)$, from $p_2 = 0.5974$ to $p_2 = 0.6181$ while keeping agent type b the same as initially. To be precise, we move 0.064 fraction (6.4%) of type a. This will cause the consistency constraint at the new candidate price $p_2 = 0.6181$ be slightly violated, i.e., $\sum_h \alpha^h \sum_{\mathbf{c}_1, k, \boldsymbol{\tau}, \Delta} x^h \left(\mathbf{c}_1, k, \boldsymbol{\tau}, p_2 = 0.6181, \Delta\right) \Delta = 0.0101$, and, respecting population proportions that need to add to unity, the consistency constraints at $p_2 = 0.5974$ is slightly violated as well, i.e., $\sum_h \alpha^h \sum_{\mathbf{c}_1, k, \boldsymbol{\tau}, \Delta} x^h \left(\mathbf{c}_1, k, \boldsymbol{\tau}, p_2 = 0.5974, \Delta\right) \Delta = -0.0096$. The utility levels each agent type a received in the second period from being in each exchange are $V^a \left(k^a, p_2 = 0.5974\right) = -1.3652$ and $V^a \left(k^a, p_2 = 0.6181\right) = -1.3591$, thus the movers of type a benefit. No type b has moved and has utility V^b as before. From the perspective of a type a chosen at random according to the fraction of movers, expected utility would increase 0.0061 units of utility and the λ's weighted objective function (16) with $\lambda^1 = 0.778$ and $\lambda^2 = 0.222$ as in Table 2, under the x notation, would increased. But of course this new allocation is not feasible as constraints are violated.

This example provides some intuition for why inactive markets are associated with binding constraints. If the constraints were not binding, they could be ignored in the maximization problem, but then the planner could reassign populations and increase the value of the objective function.[11]

One final key point: under the assignment x^h notation from (23) the condition $\delta^h \left(p_2\right) = \delta \left(p_2\right)$ in (14) is no longer needed. The fraction of a given type assigned to distinct p_2 exchanges can be different across types so long as weighted excess demands sum to zero in any active p_2 exchange

[11] Indeed, the increment in the objective function is related to the Lagrange multiplier on a binding constraint. The incremental value here is from an arbitrary selection of constraints, whereas in reality the computation of the value of Lagrange multipliers for problems with multiple equality constraints can be nontrivial. There is a computer science literature on this, solving a systems of simultaneous equations or using an algorithm that is trying out various infeasible solutions violating multiple constraints simultaneously, iterating over Lagrange multiplier successively, and then holding them fixed, iterating over the controls. For us, in the end, we are solving linear programs, so we get Lagrange multipliers from the dual problem at the optimized solution.

and fractions with a given type over p-exchanges sum to that type's mass. Indeed, to repeat, the formulation with fraction x^h is the one we use to prove the second welfare theorem in Appendix A.

2.4 The Decentralization with Rights to Trade Δ^h At Prices P_Δ

It is now a straightforward step to decentralize the market maker problem in the previous section with the $\delta^h(p_2)$ notation. Let p_1 denote the price of good z in period $t=1$, p_2 denote the price of good z in period $t=2$, and $P_\Delta(p_2)$ denote the key price of the rights. All prices are taken as given. Let the type h choice of exchange p_2 be described by indicators $\delta^h(p_2)$ and type h commodity point be $\mathbf{x}^h = \left[\mathbf{c}_1^h(p_2), k^h(p_2), \boldsymbol{\tau}^h(p_2), \delta^h(p_2), \Delta^h\left(k^h(p_2), p_2\right)\right]_{p_2}$, including rights and excess demands. The excess demand Δ^h as function of p_2 and k^h is known and given but depends on the choice of k^h, which is endogenous. These Δ^h will also play the role of rights to enter designated exchanges at $t=2$ and can in principle limit trade there. The price of rights will correspond to the shadow price of rights in the market maker problem for consistency constraint (15).

Trades are sequential over time. The initial endowments e_{w1}^h and e_{z1}^h at $t=1$ are sold at consumption prices 1 and p_1 for goods w and z, respectively, regardless of the p_2 exchange chosen. Then, in that chosen exchange, the rights $\Delta^h\left(k^h(p_2), p_2\right)$ with the indicated savings $k^h(p_2)$ determine the participation fee. Saving and consumption $\mathbf{c}_1^h(p_2)$ are then purchased by type h in the $t=1$ spot market. We then move to the corresponding p_2 spot market at $t=2$. Those rights Δ^h will be equal to what agent h will want to do at $t=2$ since the rights are excess demands, i.e., $\Delta^h\left(k^h(p_2), p_2\right) \equiv \tau_{w2}^{h*}\left(k^h(p_2), p_2\right)$. So the solution is time consistent.[12]

Definition 3. A competitive equilibrium with rights to trade is a specification of allocation $\left[\mathbf{x}^h\right]_h$, price of good z at $t=1$, p_1, spot prices p_2 for active and potential spot markets at $t=2$, and the prices of the rights to trade $[\mathbf{P}_\Delta(p_2)]_{p_2}$ such that (i) for any agent type h as a price taker, $\left[\mathbf{x}^h(p_2)\right]_{p_2}$ solves

$$\max_{\mathbf{x}^h} \sum_{p_2} \delta^h(p_2)\left[u^h\left(c_{w1}^h(p_2), c_{z1}^h(p_2)\right) + u^h\left(e_{w2}^h + \tau_{w2}^h(p_2), e_{z2}^h + Rk^h(p_2) + \tau_{z2}^h(p_2)\right)\right]$$

[12] Equivalently, we could require that all trade in each of the exchanges be done with a central counter party, CCP, who becomes the buyer for every seller and the seller for every buyer. The CCP as a broker-dealer has to make sure that all trades clear and that saving and consumptions at $t=1$ are funded. The continuum agent assumption removes any uncertainty. This equivalent formulation is useful when we have multiple active exchanges with the mixtures, as presented in a numerical example in Kilenthong and Townsend (2014).

subject to the budget constraints in the first period, which now include prices of rights $P_\Delta(p_2)$ and demand for rights $\Delta^h\left(k^h(p_2), p_2\right)$

$$\sum_{p_2} \delta^h(p_2)\left[c_{w1}^h(p_2) + p_1\left[c_{z1}^h(p_2) + k^h(p_2)\right] + P_\Delta(p_2)\,\Delta^h\left(k^h(p_2), p_2\right)\right] \le e_{w1}^h + p_1 e_{z1}^h, \tag{24}$$

and the spot-budget constraint in period $t = 2$ (21) and the non-negative saving constraint (17) for the agent type h; (ii) market-clearing conditions (13), (18), (19), (20) hold.

Using the similar steps as in the preceding section, we can then write the necessary conditions for type h maximization are as follows.

$$p_1 = \frac{u_{z1}^h}{u_{w1}^h} = \frac{u_{z2}^h}{u_{w1}^h} R + \frac{\eta^h}{u_{w1}^h} - P_\Delta(p_2)\,\Delta_k^h\left(k^h(p_2), p_2\right), \forall h = a, b. \tag{25}$$

which is starkly similar to (22) of the planner and that is the entire point. The last term in each equation is the externality correction term that come from incorporating rights. Indeed, the two equations, (22) and (25), are identical when we match the Lagrange multipliers and prices from the planner problem and the new equilibrium using the following conditions: $P_\Delta(p_2) = \frac{\mu_\Delta(p_2)}{\mu_{w1}}$ and $\eta^h = \frac{\mu^h}{\lambda^h \alpha^h}$. Specifically, the first condition shows how to recover prices of the rights to trade in both active and inactive exchanges from the Lagrange multipliers for the consistency constraints. Note that the Pareto weights λ^h associated with the competitive equilibrium can be recovered using the following condition: $\frac{\lambda^h \alpha^h}{\mu_{w1}} = \frac{1}{\eta_{bc,1}^h}$, where μ_{w1} is the Lagrange multiplier on (18), and $\eta_{bc,1}^h$ is the Lagrange multiplier on (24). The competitive equilibrium with rights picks out one of the Pareto optimal allocation as a solution to the market maker problem at Pareto weights λ^h which do not require lump-sum taxes and transfers.

For the numerical example, the competitive equilibrium with rights to trade has *one and only one active exchange*, $p_2^{op} = 0.5974$, even though all exchanges are available in principle a priori for trade. That is, in equilibrium, both types optimally choose the same p_2-exchange at $t = 1$ and hence the same p_2 spot market with, $p_2^{op} = 0.5974$. Table 3 presents equilibrium prices/fees of rights to trade, that is $P_\Delta(p_2)$ not only for p_2^{op} but also other, different spot price levels p_2. Note again that the prices/fees of non-active spot markets are available, but facing such prices, agents do not want to trade in them.[13] Again, both types choose $p_2^{op} = 0.5974$.

[13] Prices used are based on shadow prices from the planner problem, hence true marginal costs. As in standard price theory, markets in some goods can be cleared at prices implying zero activity, as when prices at marginal costs are strictly larger than the willingness to pay. Equilibrium prices can be indeterminate in a certain range in the sense that marginal cost prices can be lowered a bit but not impact the allocation.

Table 3: Equilibrium prices of rights to trade in spot markets $P_\Delta(p_2)$ at price p_2.

	$p_2 = 0.5770$	$p_2^{op} = 0.5974$	$p_2 = 0.6181$
$P_\Delta(p_2)$	1.1383	1.2116	1.2840

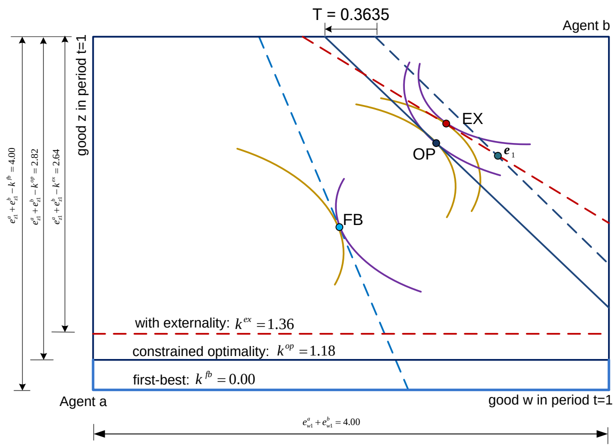
An agent type a comes into the spot market at $t = 2$ with good z in storage. So, type a buys the right to buy good w in amount $\Delta^a(p_2^{op}) = 0.2970$ (in exchange for good z, of course), where $\Delta^h(p_2^{op}) \equiv \Delta^h\left(k^h(p_2^{op}), p_2^{op}\right)$. This makes sense as agent type a is doing the saving in good z and there is too much saving in the (ex) equilibrium. On the other hand, an agent type b will be paid for her willingness to choose that market $p_2^{op} = 0.5974$. Agent type b is facing a higher price of the good z, and good z will be purchased. But there is compensation. In particular, a constrained agent $(h = b)$ with $\Delta^b(p_2^{op}) = -0.2970$ and $P_\Delta(p_2^{op}) = 1.2116$ is receiving $-P_\Delta(p_2^{op})\Delta^b(p_2^{op}) = 0.3598$ in period $t = 1$ for being in the spot market $p_2^{op} = 0.5974$. Graphically, this shifts her budget line outward at $t = 1$ by $T = 0.3598$, hence in the direction of being less constrained. This is displayed in Figure 1 along with the allocations in both dates for the competitive equilibrium with externalities (EX), the constrained optimal allocation (OP), and the first best (FB).[14]

3 General Economy

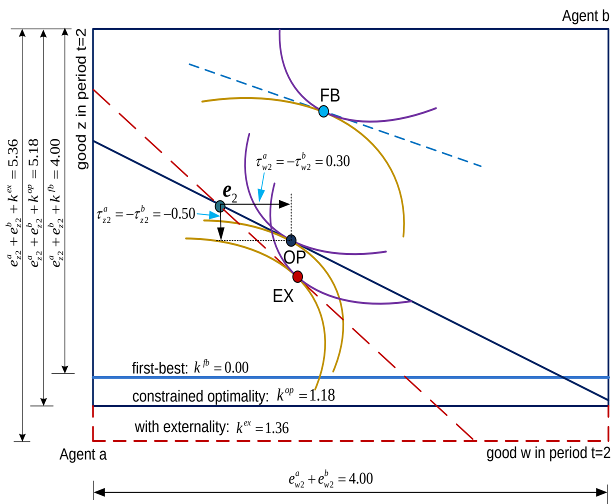
This section presents an extension of the leading example by adding uncertainty, and traded securities, yet allowing for market incompleteness and collateral constraints.[15] Consider an economy with S possible states of nature at $t = 2$, i.e., $s = 1, \ldots, S$, each of which occurs with probability π_s, $\sum_s \pi_s = 1$. Each agent type h is endowed with $\left(e^h_{w1}, e^h_{z1}\right)$ at date $t = 1$ and $\left(e^h_{w2s}, e^h_{z2s}\right)$ in state s at date $t = 2$. The utility functions u^h are strictly concave with other regularity conditions. There are J securities available for purchase or sale at $t = 1$. Let $\mathbf{D} = [D_{js}]$ be the payoff matrix of those assets at $t = 2$ where $D_{js} \in \mathbb{R}_+$ is the payoff of asset j in units of good w (the numeraire good)

[14]Trading in rights to trade generates a redistribution of wealth and welfare in general equilibrium relative to the markets without rights. Thus if nothing else were done, internalizing the externality would be beneficial to an agent type b (constrained agent) but harmful for an agent type a. To induce welfare gains for all of agents, there must be lump sum transfers, as in the second welfare theorem, which we state in Section 3.1 and prove in Appendix A.1.

[15]We provide numerical examples of an economy with active security holdings and an economy with incomplete markets in Kilenthong and Townsend (2014).



(a) Edgeworth box for $t = 1$.



(b) Edgeworth box for $t = 2$.

Figure 1: Figure 1(a) displays the allocations of goods w and z at $t = 1$. Agent a's origin moves as a function of saving; for example, with the pecuniary externality, more is saved, so points in the box reflect less consumption of good z. The EX point is achieved by movement along a budget line at equilibrium prices through the endowment. The slope is determined by the ratio of good z to good w and hence is relatively flat, with less of good z available for consumption. The OP line has a steeper slope and shifts relative to EX in favor of agent b as compensation for the type selling rights. The FB point has no saving and an equal amount of consumption for both types. Similarly, Figure 1(b) displays the allocations of goods w and z at $t = 2$. The slope of the OP line is flatter here because there is less saving carried to period $t = 2$.

in state $s = 1, 2, \ldots, S$. Here we do not include securities paying in good z as there is trade in the two goods in spot markets at price p_{2s} in terms of the numeraire so these are not needed. Let θ_j^h denote the amount of the j^{th} security acquired by an agent of type h at $t = 1$ with $\boldsymbol{\theta}^h \equiv \left[\theta_j^h\right]_j$. Here a positive number denotes the purchaser or investor, and negative the issuer, the one making the promise to deliver at $t = 2$. The collateral constraint in state s at date $t = 2$ states that there must be sufficient collateral in value to honor all promises:

$$p_{2s} R_s k^h + \sum_j D_{js} \theta_j^h \geq 0, \forall s, \tag{26}$$

where R_s is the state contingent return on the collateral. Equation (26) can be rewritten with securities $\theta_j^h > 0$ as investments with payouts D_{js} added to the value of collateral in terms of the numeraire on the left hand side and the securities $\theta_j^h < 0$ as promises with obligations D_{js} on the right hand side. This is a generalized version of $k^h \geq 0$ in the saving economy. More general obstacle-to-trade constraints applicable to our market-based approach are presented in the online Appendix F.

With potentially incomplete security markets, a given security traded at $t = 1$ has implications in general for most if not all spot prices at $t = 2$. This is one source of externalities. In addition, promises are at least potentially backed by collateral good z, which is carried over to $t = 2$, another source of externalities as in the saving example. To internalize these externalities, we thus need rights to trade indexed by the vector of spot prices $\mathbf{p} = [p_{2s}]_s$ over all states s at date $t = 2$.[16] That is, what we now term $\mathbf{p}$-exchanges must naturally deal with S spot markets as a bundle. As a result, all objects are indexed by vector $\mathbf{p}$. This is where there is a subtle difference from the saving economy. Note also that our solution is not about completing markets as illustrated in the numerical example in online Appendix E.

Let $Q_j(\mathbf{p})$ denote the price of security j at $t = 1$ executed at $t = 2$ in an exchange $\mathbf{p}$ with vector $\mathbf{Q}(\mathbf{p}) \equiv [Q_j(\mathbf{p})]_j$. The rights to trade acquired an exchange $\mathbf{p}$, to be executed at t=2 at the designated price, is denoted by a vector of rights $\boldsymbol{\Delta}^h(\mathbf{p}) \equiv \left[\Delta_s^h(\mathbf{p})\right]_s$, where for the component at state s,

$$\Delta_s^h(\mathbf{p}) \equiv \tau_{w2s}^{h*}\left(p_s, k^h(\mathbf{p}), \boldsymbol{\theta}^h(\mathbf{p})\right), \forall s, \tag{27}$$

which is the standard excess demand for good w in the spot market s at $t = 2$ for an agent type h holding collateral $k^h(\mathbf{p})$, securities $\boldsymbol{\theta}^h(\mathbf{p})$, and being in an exchange $\mathbf{p}$. For brevity, we write this

[16] Henceforth, bold typeface refers to a vector.

right $\Delta_s^h(\mathbf{p})$ as a function of the spot prices $\mathbf{p}$ only on the left hand side of (27) even though the excess demand depends on the pre-trade position coming from collateral/savings and securities. Let $\mathbf{x}^h = \left[\mathbf{c}_1^h(\mathbf{p}), k^h(\mathbf{p}), \boldsymbol{\tau}^h(\mathbf{p}), \boldsymbol{\theta}^h(\mathbf{p}), \delta^h(\mathbf{p}), \boldsymbol{\Delta}^h(\mathbf{p})\right]_{\mathbf{p}}$ denote a typical bundle or allocation for an agent type h, with $t=1$ consumption, saving, and security holdings, the choice $\mathbf{p}$-exchange, and rights consistent with excess demands all chosen by the planner.

The Pareto program with Pareto weights $\left[\lambda^h\right]_h$, analogous to the market maker problem (16), is defined as follows.

$$\max_{\left[\mathbf{x}^h\right]_h} \sum_{h,\mathbf{p}} \lambda^h \alpha^h \delta^h(\mathbf{p}) \left[u\big(c_{w1}^h(\mathbf{p}), c_{z1}^h(\mathbf{p})\big) + \sum_s \pi_s u\big(e_{w2s}^h + \sum_j D_{js}\theta_j^h(\mathbf{p}) + \tau_{w2s}^h(\mathbf{p}), e_{z2s}^h + R_s k^h(\mathbf{p}) + \tau_{z2s}^h(\mathbf{p})\big)\right]$$

subject to non-negative saving constraints, collateral constraints, the resource constraints for good w and good z in period $t=1$, the spot market budget constraints, the adding-up constraints for spot trades, the adding-up constraints for securities, and the consistency constraints, respectively,

$$\delta^h(\mathbf{p})\, k^h(\mathbf{p}) \geq 0, \forall h; \mathbf{p}, \tag{28}$$

$$\delta^h(\mathbf{p}) \left[p_{2s} R_s k^h(\mathbf{p}) + \sum_j D_{js}\theta_j^h(\mathbf{p})\right] \geq 0, \forall h; \mathbf{p}; s, \tag{29}$$

$$\sum_{\mathbf{p}} \sum_h \delta^h(\mathbf{p})\, \alpha^h c_{w1}^h(\mathbf{p}) = \sum_h \alpha^h e_{w1}^h, \tag{30}$$

$$\sum_{\mathbf{p}} \sum_h \delta^h(\mathbf{p})\, \alpha^h \left[c_{z1}^h(\mathbf{p}) + k^h(\mathbf{p})\right] = \sum_h \alpha^h e_{z1}^h, \tag{31}$$

$$\delta^h(\mathbf{p}) \left[\tau_{w2s}^h(\mathbf{p}) + p_{2s}\tau_{z2s}^h(\mathbf{p})\right] = 0, \forall h; s; \mathbf{p}, \tag{32}$$

$$\sum_h \delta^h(\mathbf{p})\, \alpha^h \tau_{\ell 2s}^h(\mathbf{p}) = 0, \forall \ell; s; \mathbf{p}, \tag{33}$$

$$\sum_h \delta^h(\mathbf{p})\, \alpha^h \theta_j^h(\mathbf{p}) = 0, \forall j; \mathbf{p}, \tag{34}$$

$$\sum_h \delta^h(\mathbf{p})\, \alpha^h \Delta_s^h(\mathbf{p}) = 0, \forall s; \mathbf{p}. \tag{35}$$

As illustrated in the example and formally proved in the following section, a constrained optimal allocation, a solution to the Pareto program, can be decentralized in a competitive equilibrium with rights to trade, a generalized version of the one defined in Section 2.4.

Definition 4. A competitive equilibrium with rights to trade is a specification of allocation $\left[\mathbf{x}^h\right]_h$, price of good z at $t=1$, p_1, spot prices $\mathbf{p} = [p_{2s}]_s$ for active and potential spot markets at $t=2$, and the prices of securities and the rights to trade $\left[\mathbf{Q}(\mathbf{p}), \mathbf{P}_\Delta(\mathbf{p})\right]_{\mathbf{p}}$ such that (i) for any agent type h as a price taker, $\left[\mathbf{x}^h(\mathbf{p})\right]_{\mathbf{p}}$ solves

$$\max_{\mathbf{x}^h} \sum_{\mathbf{p}} \delta^h(\mathbf{p}) \left[u^h\big(c_{w1}^h(\mathbf{p}), c_{z1}^h(\mathbf{p})\big) + \sum_s \pi_s u^h\big(e_{w2s}^h + \sum_j D_{js}\theta_j^h(\mathbf{p}) + \tau_{w2s}^h(\mathbf{p}), e_{z2s}^h + R_s k^h(\mathbf{p}) + \tau_{z2s}^h(\mathbf{p})\big)\right]$$

subject to the budget constraints in the first period, which now includes securities $\boldsymbol{\theta}^h$ at vector of prices $\mathbf{Q}$ and vector of rights $\boldsymbol{\Delta}^h$ at vector of prices $\mathbf{P}_\Delta$

$$\sum_{\mathbf{p}} \delta^h(\mathbf{p}) \left[c_{w1}^h(\mathbf{p}) + p_1 \left[c_{z1}^h(\mathbf{p}) + k^h(\mathbf{p})\right] + \mathbf{Q}(\mathbf{p}) \cdot \boldsymbol{\theta}^h(\mathbf{p}) + \mathbf{P}_\Delta(\mathbf{p}) \cdot \boldsymbol{\Delta}^h(\mathbf{p})\right] \leq e_{w1}^h + p_1 e_{z1}^h, \tag{36}$$

the spot-budget constraint in period $t = 2$ (33), and the collateral constraints (29); (ii) market-clearing conditions (30), (31), (33), (34), (35) hold.

3.1 Welfare Theorems and Existence Theorem

By a suitable extension of the commodity space that allows mixture representations as formalized in the appendix, the economy becomes a well-defined convex economy, i.e., the commodity space is Euclidean, the consumption sets are compact and convex, and the utility functions are linear. As a result, the first and second welfare theorems hold, and a competitive equilibrium exists.

For the first welfare theorem, the standard proof-by-contradiction argument is used. See the online Appendix A for the proof. We also assume that there is a nonsatiation point in the consumption set. Based on this non-satiation assumption, we have the formal statement:

Theorem 1. *With non-satiation of preferences, a competitive equilibrium with rights to trade in* $\mathbf{p}$*-exchanges is constrained Pareto optimal.*

The second welfare theorem can be established by matching first-order conditions of individual's and planner's problems. Though this theorem deals with any constrained Pareto optimal allocation, one of them corresponds to the competitive equilibrium without transfers. The standard proof applies. Any constrained optimal allocation can be decentralized as a compensated equilibrium. Then, use a standard cheaper-point argument (see Debreu, 1954) to show that any compensated equilibrium is a competitive equilibrium with transfers. See Appendix A.1. The formal statement:

Theorem 2. *Any constrained Pareto optimal allocation corresponding with strictly positive Pareto weights* $\lambda^h > 0, \forall h$ *can be supported as a competitive equilibrium with rights to trade with transfers.*

Finally we have the existence theorem. We use Negishi's mapping method (Negishi, 1960). The proof benefits from the second welfare theorem, that the solution to the Pareto program is a competitive equilibrium with transfers. We then show that a fixed-point of the mapping exists and represents a competitive equilibrium without transfers and, using the mapping, is constrained optimal. See the online Appendix A for the proof. The formal statement:

Theorem 3. *With local non-satiation of preferences and positive endowments, a competitive equilibrium with rights to trade exists.*

In addition, we can show that in a classical economy without pecuniary externalities, the set of competitive equilibrium allocations does not change when markets for rights to trade are introduced. See more details in the online Appendix D.

The goal of the paper was to examine if the solution to the planner problem as a constrained-optimal target allocation can be achieved in competitive markets, in particular whether we can do this for pecuniary externalities. We answered affirmatively. In the context of the featured example savings economy, one can work backwards, take our market-determined solution, and then interpret it as a tax on saving, along with lump-sum taxes and redistributions based on ownership of endowments, as in the online Appendix C. However, this can be misleading; one might draw the wrong lessons. First, if one gives the planner the power to redistribute wealth arbitrarily, one can violate the no-borrowing constraint and hence violate a key constraint on the planner problem. Second, with non-homothetic utility and incomplete securities which do not span the space of returns, tax schedules with rebates can have high dimensionality, as a function of the number of securities, and are complicated, as prices are not monotonic and move in subtle ways with type specific pre-trade positions. In contrast, our market-based solution with rights deals directly in the space of prices, much in the spirit of a reduced form or sufficient statistic argument.

4 Conclusion

Our solution concept extends to many other well-known environments in the literature that have prices in constraints beyond the role of prices in budget constraints. The collateral constraints are featured in the general model, but more generally there are sets of obstacle-to-trade constraints which include as arguments not only consumption, securities, spot trades, and inputs and outputs from production, but also vectors of prices. In the online Appendix F, we write out these constraints for additional prototype economies mentioned in the introduction.

It is natural to ask how the markets we have described would come about and how prices would be determined. Our answer is two-fold.

First, the features of institutions that our model requires are already out there and in use, in other contexts. Securities are held, maintained, and registered on electronic book entry systems and direct transfers of securities are made through specified utilities. Further, it is not uncommon that only some agents are allowed to participate, so, the necessary exclusivity required by the theory is not hard to imagine.

Secondly, our methods for proofs of the existence of Walrasian competitive equilibrium and the welfare theorems consist of converting the underlying economy with collateral, spot and forward markets, and rights to the notation of the standard Arrow, Debreu, McKenzie general equilibrium model. Thus, with that notation as a starting point, one can as in Townsend (1983) have broker dealers as intermediaries, market makers who call out prices for the commodity points and compete for the right to engage in exclusive trade with clients, buying and selling, potentially taking net positions themselves. With a continuum of traders this results in the competitive equilibrium.[17]

Potential difficulties that will have to be thought through include incomplete enumeration of future states, in which case we hope our solution works as an approximation. Related, Jeremy Stein has written about ex ante fees, a price based mechanism, for the use of a central bank credit liquidity facility, which might be contingent on some adverse states such as financial shocks, when liquidity is at a premium. Another difficulty would be vested interests that resist market reform without compensation, though this issue is not new nor peculiar to the situation here. Finally, there could be a problem with inactive exchanges. The theory requires that traders can choose any $\mathbf{p}$-exchange they want, and we do not want the inactive ones to be eliminated prematurely. Ex ante we do not know which exchanges these will be. As Stein noted, the use of rights priced with fees in financial markets can help deal with situations in which even well informed regulators cannot know the exact requirements that might be needed.[18]

References

Acemoglu, D. and A. Simsek (2012). Moral Hazard and Efficiency in General Equilibrium with Anonymous Trading. *Working Paper*.

[17] There is a related literature on the implementation of Walrasian equilibria as the outcome of market making games as in Dubey (1982), and a mechanism design literature (e.g., Allen and Jordan, 1998) regarding a minimum communication system to implement the Walrasian allocation which suggests that prices are not enough. This is also related to a computer science literature for computational algorithms that achieve the Walrasian equilibria, as in Echenique and Wierman (2011), consulting the excess demand oracle a judiciously limited number of time.

[18] See the article at https://www.federalreserve.gov/newsevents/speech/stein20130419a.htm.

Allen, B. E. and J. S. Jordan (1998). The existence of rational expectations equilibrium: A retrospective. In *Organizations with Incomplete Information: Essays in Economic Analysis, A Tribute to Roy Radner*, pp. 42–60. Cambridge University Press.

Arrow, K. (1964). The Role of Securities in the Optimal Allocation of Risk Bearing. *Review of Economic Studies 31*(2), 91–96.

Arrow, K. (1969). The Organization of Economic Activity: Issues Pertinent to the Choice of Market versus Nonmarket Allocation. *The Analysis and Evaluation of Public Expenditure: The PPB System 1*, 59–73.

Coase, R. H. (1960). The Problem of Social Cost. *Journal of Law and Economics 3*, 1.

Debreu, G. (1954). Valuation Equilibrium and Pareto Optimum. *Proceedings of the National Academy of Sciences of the United States of America 40*(7), 588.

Diamond, D. and P. Dybvig (1983). Bank Runs, Deposit Insurance, and Liquidity. *Journal of Political Economy 91*(3), 401.

Dubey, P. (1982). Price-Quantity Strategic Market Games. *Econometrica*, 111–126.

Echenique, F. and A. Wierman (2011). Finding a Walrasian Equilibrium is Easy for a Fixed Number of Agents. *Working Paper*.

Geanakoplos, J. and H. Polemarchakis (1986). Existence, Regularity and Constrained Suboptimality of Competitive Allocations when the Asset Market is Incomplete. *Uncertainty, Information and Communication: Essays in Honor of KJ Arrow 3*, 65–96.

Gorton, G. and A. Metrick (2012). Securitized Banking and the Run on Repo. *Journal of Financial Economics 104*(3), 425–451.

Greenwald, B. and J. Stiglitz (1986). Externalities in Economies with Imperfect Information and Incomplete Markets. *Quarterly Journal of Economics 101*(2), 229–264.

Hart, O. D. and L. Zingales (2013). Inefficient Provision of Liquidity. Technical report, Working Paper.

Jacklin, C. (1987). Demand Deposits, Trading Restrictions, and Risk Sharing. *Contractual Arrangements for Intertemporal Trade*, 26–47.

Kilenthong, W. T. and R. M. Townsend (2011). Information-Constrained Optima with Retrading: An Externality and its Market-Based Solution. *Journal of Economic Theory 146*(3), 1042–1077.

Kilenthong, W. T. and R. M. Townsend (2014). Segregated Security Exchanges with Ex Ante Rights to Trade. Technical report, National Bureau of Economic Research.

Lindahl, E. (1958). Just Taxation–A Positive Solution. In R. A. Musgrave and A. T. Peacock (Eds.), *Classics in the theory of public finance*. Macmillan, London.

Lorenzoni, G. (2008). Inefficient Credit Booms. *Review of Economic Studies 75*(3), 809–833.

Mas-Colell, A., M. D. Whinston, and J. R. Green (1995). *Microeconomic Theory*, Volume 1. Oxford university press New York.

Meade, J. E. (1952). External Economies and Diseconomies in a Competitive Situation. *The Economic Journal*, 54–67.

Negishi, T. (1960). Welfare Economics and Existence of an Equilibrium for a Competitive Economy. *Metroeconomica 12*(2-3), 92–97.

Prescott, E. C. and R. M. Townsend (1984a). General Competitive Analysis in an Economy with Private Information. *International Economic Review 25*(1), 1–20.

Prescott, E. C. and R. M. Townsend (1984b). Pareto Optima and Competitive Equilibria with Adverse Selection and Moral Hazard. *Econometrica 52*(1), 21–45.

Townsend, R. M. (1983). Theories of intermediated structures. *Carnegie-Rochester Conference Series on Public Policy 18*, 221–272.

A Proofs

At the individual level, for each agent type h, let $x^h\left(\mathbf{c}_1, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{p}, \boldsymbol{\Delta}\right) \geq 0$ denote the probability of receiving period $t = 1$ consumption $\mathbf{c}_1$, collateral k, securities $\boldsymbol{\theta}$, period $t = 2$ spot trades $\boldsymbol{\tau}$, and being in exchanges indexed by $\mathbf{p} \equiv [p_{2s}]_s$ with rights to trade $\boldsymbol{\Delta}$. We write again the spot market budget, the non-negative saving constraint and the collateral constraints in state s at date $t = 2$:

$$\tau_{w2s} + p_{2s}\tau_{z2s} = 0, \forall s;\;\; k \geq 0;\;\; p_{2s}R_s k + \sum_j D_{js}\theta_j \geq 0, \forall s. \tag{37}$$

Accordingly, we impose the following condition on a probability measure:

$$x^h(\mathbf{c}_1, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{p}, \boldsymbol{\Delta}) \geq 0 \text{ if } (\mathbf{c}_1, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{p}, \boldsymbol{\Delta}) \text{ satisfies (27), (37),} \tag{38}$$

and zero otherwise. The consumption possibility set of an agent type h is defined by

$$X^h = \left\{ \mathbf{x}^h \in \mathbb{R}^n_+ : \sum_{\mathbf{c}_1, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{p}, \boldsymbol{\Delta}} x^h(\mathbf{c}_1, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{p}, \boldsymbol{\Delta}) = 1, \text{ and (38) holds} \right\}. \tag{39}$$

Note that X^h is compact and convex. In addition, the non-emptiness of X^h is guaranteed by assigning mass one to each agent's endowment, i.e., no trade is a feasible option. For notational purposes, let $\mathbf{w} \equiv (\mathbf{c}_1, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{p}, \boldsymbol{\Delta})$ be a typical bundle, and the utility derived from it for an agent type h is defined by $U^h(\mathbf{w}) = u^h(c_{w1}, c_{z1}) + \sum_s \pi_s u^h\big(e^h_{w2s} + \sum_j D_{js}\theta^h_j + \tau_{w2s}, e^h_{z2s} + R_s k + \tau_{z2s}\big)$. Then, we have the maximization problem for agents as part of the definition of equilibrium: for each h, $\mathbf{x}^h \in X^h$ solves

$$\max_{\mathbf{x}^h \in X^h} \sum_{\mathbf{w}} x^h(\mathbf{w}) U^h(\mathbf{w}) \tag{40}$$

subject to $\mathbf{x}^h \in X^h$, and period $t = 1$ budget constraint, that the valuation of endowments sold provides revenue for purchase of the lotteries.

$$\sum_{\mathbf{w}} P(\mathbf{w}) x^h(\mathbf{w}) \quad \leq \quad e^h_{w1} + p_1 e^h_{z1}, \tag{41}$$

taking price of good z at $t = 1$, p_1, and prices of lottery, $P(\mathbf{w})$ as given.

We introduce broker dealers that run the $\mathbf{p}$-exchanges and deal with households for trades in securities, collateral, rights to trade and spot trades. The consumption $\mathbf{c}_1$ and collateral k commitments are sold but must be funded by the requisite amount of consumption goods and collateral. Securities, rights and spot trades do not require resources but are cleared by the broker-dealers. There are constant returns to scale in these activities so it is as if there were one representative broker-dealer. Let $b(\mathbf{c}_1, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{p}, \boldsymbol{\Delta})$ denote the quantity of commitment to provide $(\mathbf{c}_1, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{p}, \boldsymbol{\Delta})$. See Prescott and Townsend (1984a) for the introduction of broker-dealer. The broker-dealer takes prices p_1 and $P(\mathbf{w})$ as given and supplies $\mathbf{b}$ to solve the following profit maximization problem:

$$\max_{\mathbf{b}} \sum_{\mathbf{w}} b(\mathbf{w}) \left[P(\mathbf{w}) - c_{w1} - p_1 c_{z1} - p_1 k \right] \tag{42}$$

subject to clearing constraints:

$$\sum_{\mathbf{c}_1, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \boldsymbol{\Delta}} b(\mathbf{c}_1, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{p}, \boldsymbol{\Delta}) \theta_j = 0, \ \forall j; \mathbf{p}, \tag{43}$$

$$\sum_{\mathbf{c}_1, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \boldsymbol{\Delta}} b(\mathbf{c}_1, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{p}, \boldsymbol{\Delta}) \tau_{\ell 2s} = 0, \ \forall s; \ell; \mathbf{p}, \tag{44}$$

$$\sum_{\mathbf{c}_1,k,\boldsymbol{\theta},\boldsymbol{\tau},\boldsymbol{\Delta}} b\left(\mathbf{c}_1,k,\boldsymbol{\theta},\boldsymbol{\tau},\mathbf{p},\boldsymbol{\Delta}\right)\Delta_s = 0\ ,\forall s;\mathbf{p}. \tag{45}$$

Market clearing conditions in the two consumption goods is standard, purchased consumptions and collateral by the broker-dealer equals supply of endowments from the households:

$$\sum_{\mathbf{w}} b\left(\mathbf{w}\right)c_{w1} = \sum_h \alpha^h e^h_{w1}, \tag{46}$$

$$\sum_{\mathbf{w}} b\left(\mathbf{w}\right)\left[c_{z1}+k\right] = \sum_h \alpha^h e^h_{z1}. \tag{47}$$

The net demand for contracts by households, allowing non-degenerate fractions in the population, equals the supply of contracts by the broker-dealer:

$$\sum_h \alpha^h x^h\left(\mathbf{w}\right) = b\left(\mathbf{w}\right),\ \forall \mathbf{w}. \tag{48}$$

See Kilenthong and Townsend (2014) for a particular clarified example of what broker-dealers in the context of an environment with multiple active exchanges.

Definition 5. A competitive equilibrium with rights to trade (with mixtures) is a specification of allocation $\left(\mathbf{x}^h,\mathbf{b}\right)$, and prices $\left(p_1,P\left(\mathbf{w}\right)\right)$ such that

(i) for each h, $\mathbf{x}^h \in X^h$ solves the utility maximization problem (40) taking prices as given;

(ii) for the broker-dealer, $\mathbf{b}$ solves the maximization problem (42), taking prices as given;

(iii) market clearing conditions (46)-(48) hold.

The Pareto problem with Pareto weights $\left[\lambda^h\right]_h$ is defined as follows.

$$\max_{\left[\mathbf{x}^h\in X^h\right]_h} \sum_h \lambda^h\alpha^h \sum_{\mathbf{w}} x^h\left(\mathbf{w}\right)U^h\left(\mathbf{w}\right) \tag{49}$$

subject to

$$\sum_h \alpha^h \sum_{\mathbf{w}} x^h\left(\mathbf{w}\right)c_{w1} = \sum_h \alpha^h e^h_{w1}, \tag{50}$$

$$\sum_h \alpha^h \sum_{\mathbf{w}} x^h\left(\mathbf{w}\right)\left[c_{z1}+k\right] = \sum_h \alpha^h e^h_{z1}, \tag{51}$$

$$\sum_h \alpha^h \sum_{\mathbf{c}_1,k,\boldsymbol{\theta},\boldsymbol{\tau},\boldsymbol{\Delta}} x^h\left(\mathbf{c}_1,k,\boldsymbol{\theta},\boldsymbol{\tau},\mathbf{p},\boldsymbol{\Delta}\right)\tau_{\ell 2s} = 0,\forall \ell;s;\mathbf{p}, \tag{52}$$

$$\sum_h \alpha^h \sum_{\mathbf{c}_1,k,\boldsymbol{\theta},\boldsymbol{\tau},\boldsymbol{\Delta}} x^h\left(\mathbf{c}_1,k,\boldsymbol{\theta},\boldsymbol{\tau},\mathbf{p},\boldsymbol{\Delta}\right)\theta_j = 0,\forall j;\mathbf{p}, \tag{53}$$

$$\sum_h \alpha^h \sum_{\mathbf{c}_1,k,\boldsymbol{\theta},\boldsymbol{\tau},\boldsymbol{\Delta}} x^h\left(\mathbf{c}_1,k,\boldsymbol{\theta},\boldsymbol{\tau},\mathbf{p},\boldsymbol{\Delta}\right)\Delta_s = 0,\forall s;\mathbf{p}. \tag{54}$$

A.1 Proof of The Second Welfare Theorem

Proof of Theorem 2. Since the optimization problems are well-defined concave problems, Kuhn-Tucker conditions are necessary and sufficient. The proof is divided into three steps.

(i) Kuhn-Tucker conditions for a compensated equilibrium allocation: Let $\hat{\gamma}_U^h$ and $\hat{\gamma}_l^h$ be the Lagrange multiplier for the reservation-utility constraint, and for the probability constraint, respectively. The optimal condition for $x^h(\mathbf{w})$ is given by

$$\hat{\gamma}_U^h U^h(\mathbf{w}) \leq P(\mathbf{w}) + \hat{\gamma}_l^h, \tag{55}$$

where the inequality holds with equality if $x^h(\mathbf{w}) > 0$. The optimal condition for the broker-dealer's profit maximization problem implies that, for any typical bundle $\mathbf{w}$,

$$P(\mathbf{w}) \leq c_{w1} + p_1 [c_{z1} + k] + \sum_j \widehat{Q}_j(\mathbf{p})\,\theta_j + \sum_s \sum_\ell \hat{p}_\ell(\mathbf{p}, s)\,\tau_{\ell 2s} + \sum_s \widehat{P}_\Delta(\mathbf{p}, s)\,\Delta_s,$$

where $\widehat{Q}_j(\mathbf{p})$, $\hat{p}_\ell(\mathbf{p}, s)$ and $\widehat{P}_\Delta(\mathbf{p}, s)$ are the Lagrange multipliers for constraints (43)-(45). The condition holds with equality if $b(\mathbf{w}) > 0$.

(ii) Kuhn-Tucker conditions for Pareto optimal allocations: A solution to the Pareto program satisfies the following optimal condition

$$\lambda^h U^h(\mathbf{w}) \leq \tilde{p}_{w1} c_{w1} + \tilde{p}_{z1} [c_{z1} + k] + \sum_j \widetilde{Q}_j(\mathbf{p})\,\theta_j + \sum_s \sum_\ell \tilde{p}_\ell(\mathbf{p}, s)\,\tau_{\ell 2s} + \sum_s \widetilde{P}_\Delta(\mathbf{p}, s)\,\Delta_s + \tilde{\gamma}_l^h,$$

where $\tilde{\gamma}_l^h$ is the Lagrange multiplier for the probability constraint, and $\tilde{p}_{w1}$, $\tilde{p}_{z1}$, $\widetilde{Q}_j(\mathbf{p})$, $\tilde{p}_\ell(\mathbf{p}, s)$ and $\widetilde{P}_\Delta(\mathbf{p}, s)$ are the Lagrange multipliers for constraints (50)-(54), respectively. Again, the condition holds with equality if $x^h(\mathbf{w}) > 0$.

(iii) Matching dual variables and prices: We can now set $\hat{\gamma}_U^h = \frac{\lambda^h}{\tilde{p}_{w1}}$, $p_1 = \frac{\tilde{p}_{z1}}{\tilde{p}_{w1}}$, $\widehat{Q}_j(\mathbf{p}) = \frac{\widetilde{Q}_j(\mathbf{p})}{\tilde{p}_{w1}}$, $\hat{p}_\ell(\mathbf{p}, s) = \frac{\tilde{p}_\ell(\mathbf{p}, s)}{\tilde{p}_{w1}}$ and $\widehat{P}_\Delta(\mathbf{p}, s) = \frac{\widetilde{P}_\Delta(\mathbf{p}, s)}{\tilde{p}_{w1}}$, and $\hat{\gamma}_l^h = \frac{\tilde{\gamma}_l^h}{\tilde{p}_{w1}}$. These matching conditions imply that the optimal conditions of the Pareto program are equivalent to the optimal conditions for consumers' and broker-dealer's problems in the compensated equilibrium. To sum up, any Pareto optimal allocation is a compensated equilibrium.

We can show that any compensated equilibrium, corresponding to $\lambda^h > 0$, is a competitive equilibrium with transfers using the cheaper point argument, which is obvious given the strictly positive Pareto weight and strictly positive endowment. Using the cheaper-point argument, a compensated equilibrium is a competitive equilibrium with transfers. □