14.04 Intermediate Micro Theory: Lecture 19

Income and Substitution Effects

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Demand Functions

- Solving the consumer utility maximization problem yields a set of individual demand function

$$\begin{aligned} X_1^* &= d_1(p_1, p_2, ..., p_n, I) \\ X_2^* &= d_2(p_1, p_2, ..., p_n, I) \\ &\vdots \\ X_n^* &= d_n(p_1, p_2, ..., p_n, I) \end{aligned}$$

  Here d_k denotes an individual's demand function for good k, p_k denotes the price of good k, and I denotes the individual's income.
- Goal today: Characterize how individual demand functions respond to changes in income and prices

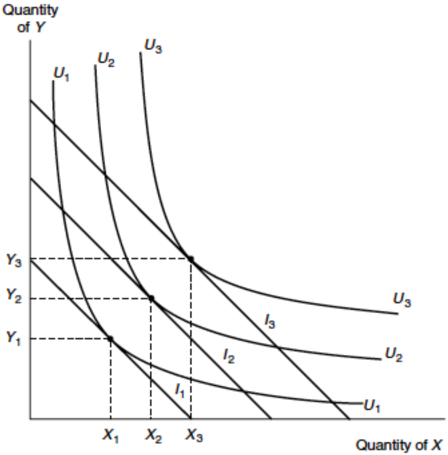
Homogeneity

- If we were to double all prices and income, the optimal quantities demanded would remain unchanged. Indeed,

$$X_k^* = d_k\left(tp_1, tp_2, ..., tp_n, tI\right) = d_k\left(p_1, p_2, ..., p_n, I\right)$$

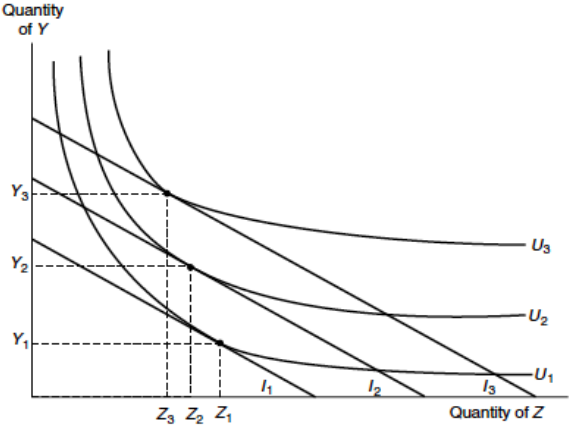
  for any $t > 0$
- In other words, individual demand is homogeneous of degree zero in all prices and income.

Effect of an Increase in Income



As income increases from I_1 to I_2 to I_3, the optimal (utility-maximizing) choices of X and Y are shown by the successively higher points of tangency. Notice that the budget constraint shifts in a parallel way because its slope (given by $-P_X/P_Y$) does not change.

An Indifference Curve Map Exhibiting Inferiority



In this diagram, good Z is inferior because the quantity purchased actually declines as income increases. Y is a normal good (as it must be if there are only two goods available), and purchases of Y increase as total expenditures increase.

Notice that indifference curves do not have to be “oddly” shaped to exhibit inferiority; the curves corresponding to goods Y and Z continue to obey the assumption of a diminishing MRS. Good Z is inferior because of the way it relates to the other goods available (good Y here), not because of a peculiarity unique to it.

Normal and Inferior Goods

- A good X_i for which $\partial X_i / \partial I < 0$ over some range of income changes is an **inferior** good in that range
- If $\partial X_i / \partial I \geq 0$ over some range of income variation, the good is a **normal**, or “noninferior,” good in that range

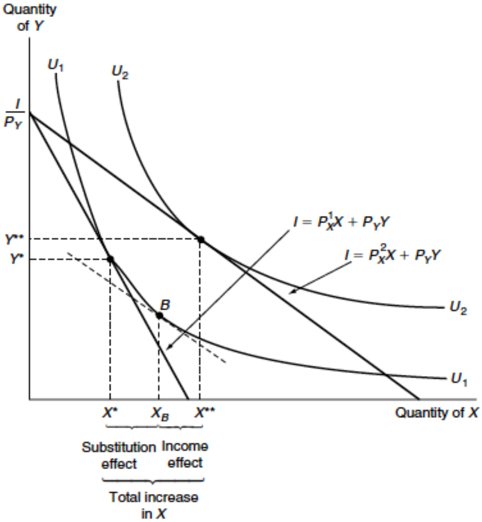
Engel's Law

	Annual Income		
Expenditure Item	**\$225–\$300**	**\$450–\$600**	**\$750–\$1,000**
Food	62.0%	55.0%	50.0%
Clothing	16.0	18.0	18.0
Lodging, light, and fuel	17.0	17.0	17.0
Services (education, legal, health)	4.0	7.5	11.5
Comfort and recreation	1.0	2.5	3.5
Total	100.0%	100.0%	100.0%

SOURCE: Adapted from A. Marshall, *Principles of Economics*, 8th ed. (London: Macmillan, 1920), p. 97.

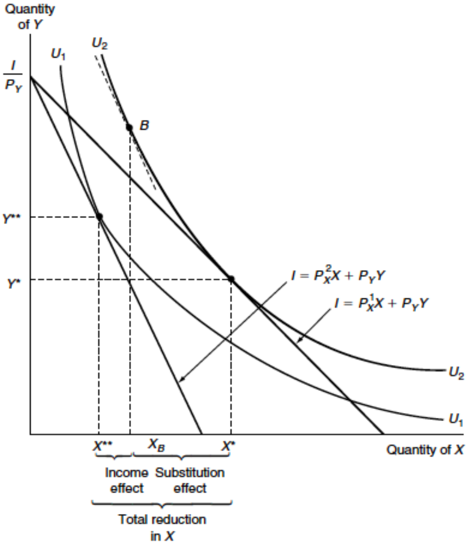
Engel drew what was perhaps the first empirical generalization about consumer behavior: The proportion of total expenditure devoted to food declines as income rises. In other words, food is a necessity whose consumption rises less rapidly than does income. That hypothesis has come to be known as "Engel's law," and it has been verified in hundreds of studies.

Income and Substitution Effects of a Price Decrease



When the price of X falls from P_X^1 to P_X^2, the utility-maximizing choice shifts from X*, Y* to X**, Y**. This movement can be broken down into two analytically different effects: first, the substitution effect, involving a movement along the initial indifference curve to point B, where the MRS is equal to the new price ratio; and secondly, the income effect, entailing a movement to a higher level of utility, because real income has increased. In the diagram, both the substitution and income effects cause more X to be bought when its price declines. Notice that point I/P_Y is the same as before the price change. This is because P_Y has not changed. Point I/P_Y therefore appears on both the old and new budget constraint.

Income and Substitution Effects of a Price Increase

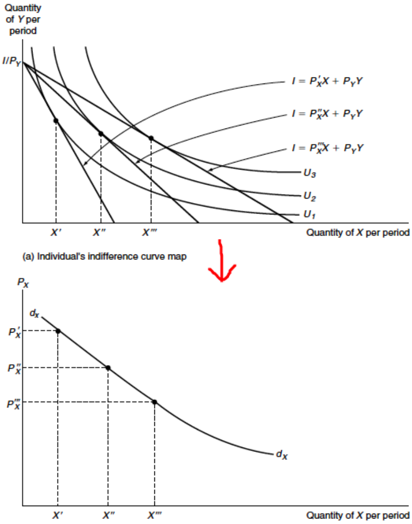


When the price of X increases, the budget constraint shifts inward. The movement from the initial utility-maximizing point (X*, Y *) to the new point (X**, Y **) can be analyzed as two separate effects. The substitution effect would be depicted as a movement to point B on the initial indifference curve (U2). The price increase, however, would create a loss of purchasing power and a consequent movement to a lower indifference curve. This is the income effect. In the diagram, both the income and substitution effects cause the quantity of X to fall as a result of the increase in its price. Again, the point I/PY is not affected by the change in the price of X.

Giffen's Paradox

- If the income effect of a price change is strong enough, the change in price and the resulting change in the quantity demanded could actually move in the same direction.
- Legend has it that the English economist Robert Giffen observed this paradox in nineteenth-century Ireland—when the price of potatoes rose, people reportedly consumed more of them. This peculiar result can be explained by looking at the size of the income effect of a change in the price of potatoes.
- Potatoes were not only inferior goods, but also used up a large portion of the Irish people's income. An increase in the price of potatoes therefore reduced real income substantially. The Irish were forced to cut back on other luxury food consumption in order to buy more potatoes.
- Even though this rendering of events is historically implausible, the possibility of an increase in the quantity demanded in response to an increase in the price of a good has come to be known as Giffen's paradox. Later we will provide a mathematical analysis of how Giffen's Paradox can occur.

Construction of an Individual's Demand Curve



(a) Individual's indifference curve map

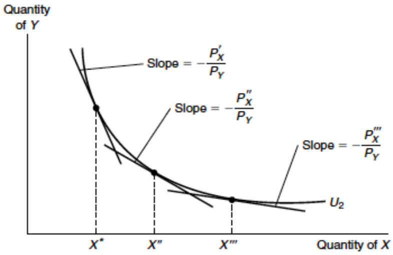
(b) Demand curve

In (a) the individual's utility maximizing choices of X and Y are shown for three different prices of X. In (b), this relationship between P_X and X is used to construct the demand curve for X. The demand curve is drawn on the assumption that P_Y, I, and preferences remain constant as P_X varies.

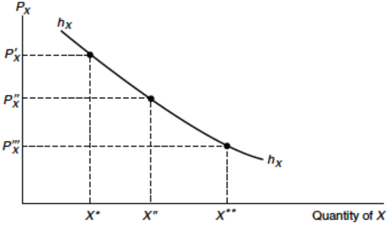
Uncompensated and Compensated Demand Curves

- An individual **uncompensated** (or **Marshallian**) demand curve shows the relationship between the price of a good and the quantity of that good purchased by an individual, assuming that all other prices and **income** are held constant.
- A **compensated** (or **Hicksian**) demand curve shows the relationship between the price of a good and the quantity purchased on the assumption that other prices and **utility** are held constant. The curve therefore illustrates only substitutions effects.

Construction of a Compensated Demand Curve



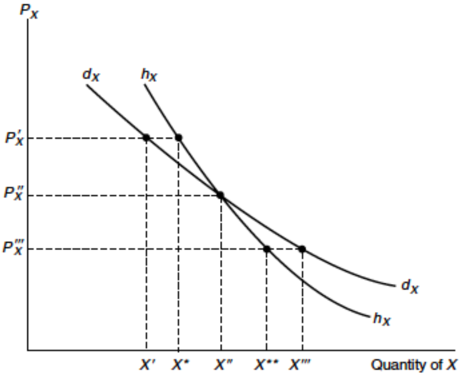
(a) Individual's indifference curve map



(b) Compensated demand curve

The curve h_X shows how the quantity of X demanded changes with P_X changes, holding P_Y and utility constant. That is, the individual's income is "compensated" so as to keep utility constant. Hence h_X reflects only substitution effects of changing prices.

Compensated vs Uncompensated Demand



The compensated (h_X) and uncompensated (d_X) demand curves intersect at P''_X because X'' is demand under each concept. For prices above P''_X, the individual's income is increased with the compensated demand curve, so more X is demanded than with the uncompensated curve. For prices below P''_X, income is reduced for the compensated curve, so less X is demanded than with the uncompensated curve. The curve d_X is flatter because it incorporates both substitution and income effects whereas the curve h_X reflects only substitution effects.

Slutsky's Equation

- The utility-maximization hypothesis shows that the substitution and income effects arising from a price can be represented by

$$\frac{\partial d_X}{\partial P_X} = \underbrace{\left.\frac{\partial X}{\partial P_X}\right|_{U=\text{constant}}}_{\text{substitution effect}} - \underbrace{X\frac{\partial X}{\partial I}}_{\text{income effect}} .$$

- This is known as Slutsky's equation

Expenditure Function

Take a consumer with a consumption set $X \subseteq \mathbb{R}_{+}^{L}$, with preferences given by a continuous utility function $u : X \to \mathbb{R}$. To simplify the analysis, assume that $X = \mathbb{R}_{+}^{L}$. The utility maximization problem (UMP) is, as we said in the previous lectures:

$$\begin{aligned} v(p, w) &\equiv \max_{x \in \mathbb{R}_{+}^{L}} u(x) \\ s.t &: px \leq w \end{aligned} \tag{1}$$

with $w \geq 0$ and $p \gg 0$, with $v(p, w)$ the indirect utility function. Now, let $u = v(p, w)$ the attained utility level for this problem. The expenditure minimization problem (EMP) is a related problem: **what is the minimum amount that needs to be expended to attain at least that level of utility?**

$$\begin{aligned} e(p, u) &\equiv \min_{x \in \mathbb{R}_{+}^{L}} px \\ &\quad s.t : u(x) \geq u \end{aligned} \tag{2}$$

The function $e(p, u)$ is called the expenditure function.

Proposition (Duality (MWG. Prop. 3.E.1))

Suppose that $u(.)$ is a continuous utility function representing a locally nonsatiated preference relation $\succsim$ defined on the consumption set $X = \mathbb{R}^L_+$ and that the price vector is $p \gg 0$. We then have that:

1. *If x^* is optimal in the (UMP) when wealth is $w > 0$, then x^* is optimal in the (EMP) when the required utility level is $u(x^*)$. Moreover, the minimized expenditure level in this (EMP) is exactly w*
2. *If x^* is optimal in the (EMP) when the required utility level is $u > u(0)$, then x^* is optimal in the (UMP) when wealth is px^*. Moreover, the maximized utility level in this (UMP) is exactly u.*

The previous proposition states that for all $(p, w) \gg 0$ and $u > u(0)$ we have that:

$$e[p, v(p, w)] = w \text{ for all } (p, w) \quad (3)$$

$$v[p, e(p, u)] = u \text{ for all } (p, u) \quad (4)$$

Where (3) comes from Claim 1, and (4) comes from Claim 2. This implies that given a price vector p, **the functions** $e(\bar{p}, .)$ **and** $v(\bar{p}, .)$ **are inverses of each other.** This result is known as duality between the expenditure minimization problem and the utility maximization problem.

Hicksian Demand

Define

$$\begin{aligned} h(p,u) &= \arg\min_{x \in \mathbb{R}_+^L} px \\ s.t &: u(x) \geq u \end{aligned}$$

as the **Hicksian demand correspondence** for the (EMP) program. Using the previous result, we can also state the relationship between the Hicksian demand $h(p,u)$ and the Walrasian demand $x(p,w)$ defined in previous lectures:

$$h[p, v(p,w)] = x(p,w) \text{ for all } (p,w) \quad (5)$$

$$x[p, e(p,u)] = h(p,u) \text{ for all } (p,u) \quad (6)$$

Equation (5) means that the expenditure minimizing bundle at value $v(p,w)$ is exactly the Walrasian Demand, and equation (6) expresses that the Walrasian demand when income is $e(p,u)$ coincides with the Hicksian demand.

The Hicksian demand is usually called the **compensated demand**: the idea comes from the following thought exercise: suppose prices change from p to p', and the demand were given by $x(p, w)$, maintaining income w fixed. Define then $w' = e[p', u]$ with $u = v(p, w)$. This level of income w' is the one that, under the new prices p', would minimize expenditures at p' necessary to achieve the exact same indirect utility $v(p, w)$. In that sense, we are compensating the agent for the change in the purchasing power of her wealth, by letting her achieve the same level of utility as before. However, that does not mean that the demand will remain constant: the new demand will be given by

$$x^* = x(p', w') = x\left[p', e(p', u)\right] \underset{(i)}{=} h(p', u) \tag{7}$$

Using in (i) condition (6). So $h(p', u)$ gives the compensated demand, once we compensate the agent for the change in real wealth (i.e. the change in the purchasing power of her wealth).

purchasing power of her wealth). In Figure 2 we see an illustration of this idea; when $L = 2$, we normalize the price of $p_1 = 1$ and we analyze what happens when the price of commodity 2 goes from p_2 to $p_2' > p_2$: the compensating wealth level in units of good 1 is $w' = w + \Delta w$ with $\Delta w = e(p', u) - w$

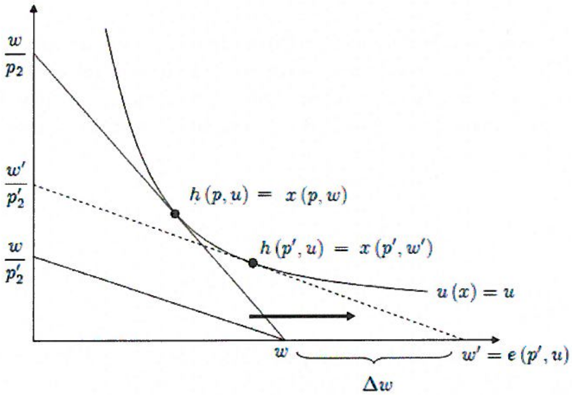


Figure 2: Compensated demand: case with $p_1 = 1$ and $p_2' > p_2$

The next two propositions gives us some basic properties of the expenditure function e and the Hicksian demand h.

Proposition (Properties of the expenditure function: MWG Prop. 3.E.2)

Suppose that $u(.)$ is a continuous utility function representing a locally nonsatiated preference relation $\succsim$ defined on the consumption set $X = \mathbb{R}^L_+$. Then, the following statements hold:

1. $e(p, u)$ *is homogeneous of degree one in* p, *and* $h(p, u)$ *is homogeneous of degree* 0 *in* p.
2. $e(p, u)$ *is strictly increasing in* u *and nondecreasing in* p_l *for any* l
3. $e(p, u)$*is concave in* p *(for fixed* u*)*
4. $e(p, u)$ *is continuous in* (p, u)

Proposition (Properties of Hicksian demand: MWG Prop. 3.E.3, 3.G.1, 3.G.2)

Suppose u is continuous and locally non-satiated. Then

1. *(No excess utility) If* $x \in h(p,u) \Longrightarrow u(x) = u$
2. *If* $\succsim$ *are convex, then* $h(p,u)$ *is convex valued. If preferences are strictly convex, then* $h(p,u)$ *is single valued and continuous.*
3. *If* $h(p,u)$ *is single valued, then* $e(p,u)$ *is differentiable, and moreover*

$$\frac{\partial e(p,u)}{\partial p_l} = h_l(p,u) \text{ for all } l = 1, 2, ..., L \tag{8}$$

4. *If h is differentiable, then the Jacobian matrix with respect to prices;* $\nabla_p h(p,u)$ *(a* $L \times L$ *matrix) satisfies:*

$$\nabla_p h(p,u) = \nabla_p^2 e(p,u) \tag{9}$$

a symmetric, negative semidefinite matrix